Introduction to Statistical Mechanics of Disordered Spin Systems

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Preface

This lecture course is devoted to the special part the statistical mechanics which deals with the classical spin systems with quenched disorder. The course is assumed to be of the pedagogical character, and it aims to make the reader to get into the subject starting from fundamentals. The course is supposed to be selfcontained (it is not required to go through all the references to understand something) being understandable for any student having basic knowledge in theoretical physics and statistical mechanics.

The first part of the course is devoted to the physics of spin-glass systems, where the quenched disorder is the dominant factor. The emphasis is made on a general qualitative description of the physical phenomena, being mostly based on the results obtained in the framework of the mean-field theory of spin-glasses with long-range interactions [1,2]. First, the general problems of the spin-glass state are discussed at the qualitative level. In Chapters 3-5 the "magic" of the replica symmetry breaking (RSB) scheme is explained in details, and the physics behind it is discussed. This part also contains the detailed derivation of the ultrametric structure of the space of the spin-glass states as well as its scaling properties. Chapter 6 is devoted to the series of recent experiments on real spin-glass materials, which on a qualitative level confirm the basic theoretical predictions.

The second part of the course is mainly devoted to theory of the critical phenomena at the phase transitions of the second order in the presence of weak quenched disorder [24]. Theory of the critical phenomena deals with macroscopic statistical systems in a close vicinity of the phase transition point where spontaneous symmetry breaking takes place, and the situation is characterized by large-scale fluctuations. According to the traditional scaling theory of the second-order phase transitions the large-scale fluctuations are characterized by certain dominant scale, or the correlation length, R_c. The correlation length grows as the critical point is approached, where it becomes infinite. The large-scale fluctuations lead to singularities in the macroscopic characteristics of the system as a whole. These singularities are the main subject of the theory. Chapter 7 is devoted to the systematic consideration of the traditional renormalization-group (RG) theory of the critical phenomena, including ϵ-expansion [25].

Originally, many years ago, it was generally believed that quenched disorder either completely destroy the long range fluctuations, such that the singularities of the thermodynamical functions are getting smoothed out, or it can produce only a shift of the critical point but cannot effect the critical behavior itself. Later it was realized that intermediate situation is also possible, in which a new critical behavior, with new universal critical exponents, is established sufficiently close to the phase transition point. In terms of the RG approach the standard procedure for obtaining a new universal "disordered" critical regime for the vector ferromagnetic spin systems is considered in Chapter 8.

However, according to the recent developments in this field, the effects of the quenched disorder on the critical behaviour could appear to be more complicated, and in certain cases completely new type of the critical phenomena of the spin-glass nature could be established in the close vicinity of the critical point. In Chapter 9 the RG theory for the vector ferromagnetic spin systems is generalized to take into account the non-perturbative spin-glass type phenomena. It is demonstrated that whenever the disorder is relevant for the critical behavior there exists no stable fixed points, and the RG flows lead to the so called strong coupling regime at a finite spatial scale. The physical consequences of the obtained RG solutions are discussed.

In Chapter 10 we consider the critical properties of the two-dimensional disordered Ising model. In terms of the Fermion fields formalism the exact solution for the critical behavior of the specific heat is derived, and the phase diagram as well as the results of the recent numerical simulations are discussed.

Finally, in Chapter 11 the Ising spin systems with quenched random fields are considered. The statistical systems of this type exhibit qualitatively different properties compared to those considered before. The random field Ising models are of special interest for two reasons. First, because they have many experimentally accessible realizations, and second, because despite extensive theoretical and experimental efforts during last twenty years very little is understood about their basic properties even at the qualitative level.

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CONTENTS:

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Part II. Critical Phenomena and Quenched Disorder

1 INTRODUCTION

1.1 General principles of the statistical mechanics

In the most simple terms the basic statements of the statistical mechanics can be introduced in the following way. Let the microscopic state of a *macroscopic* system having many degrees of freedom is described by the configurations of N variables $\{s_i\}$, $(i = 1, 2, ..., N)$. The basic quantity characterizing the microscopic states is called the *energy* H, and it is defined as a function of all the microscopic variables $\{s_i\}$:

$$H = H(s_1, s_2, ..., s_N) \equiv H[s]$$

The microscopic dynamic behavior of the system is defined by some dynamic differential equations such that, in general, the energy of the system tends to a minimum. Besides, it is assumed that no observable system can be perfectly isolated from the surrounding world, and the effect of the interaction with the surroundings (the thermal bath) is believed to produce the so called, *thermal noise* in the exact dynamical equations. The thermal (white) noise acts as random and uncorrelated fluctuations which produces the randomization and the mixing of the exact dynamical trajectories of the system.

Let $A[s]$ be some observable quantity. The quantities, which are of interest in the statistical mechanics, are the *averaged* values of the observables. In other words, instead of studying the exact evolution in time of the value $A[s(t)]$, one introduces the averaged quantity:

$$\langle A \rangle \; = \; \lim_{t \to \infty} \frac{1}{t} \int_0^t dt' A[s(t')] \tag{1.1}$$

which could be formally obtained after the observation during infinite time period.

The fundamental hypothesis of the equilibrium statistical mechanics lies in the following. It is believed that, owing to the mixing of the dynamic trajectories, after an infinitely long observation time the system in general, "visits" its different microscopic states many times, and therefore the averaged quantity in Eq.(1.1) could be obtained by averaging over the *ensemble* of the states instead of that over the time:

$$\langle A \rangle = \int ds_1 ds_2 ... ds_N A[s] P(s_1, s_2, ..., s_N) \tag{1.2}$$

Here $P[s]$ is the *probability distribution function* of the microscopic states of the system. In other words, it is believed, that because of the mixing of the dynamic trajectories, instead of solving the exact dynamics, the system could be statistically described in terms of the probabilities of its microscopic states given by the function $P[s]$. The probability distribution function, whatever it is, must be normalized:

$$\int ds_1 ds_2 ... ds_N P(s_1, s_2, ..., s_N) \; = \; 1 \tag{1.3}$$

The fundamental quantity of the statistical mechanics which characterizes the probability distribution itself is called the entropy. It is defined as the average of the logarithm of the distribution function:

$$S \; = \; -\langle \log(P[s]) \rangle \; \equiv \; -\int ds_1 ds_2 ... ds_N P[s] \log(P[s]) \tag{1.4}$$

First of all, it obvious from the above definition, that because of the normalization (1.3), the entropy is at least non-negative. In general, the value of the entropy could tell to what extent the state of the system is "ordered". Consider a simple illustrative example. Let the (discrete) microscopic states of the system be labeled by an index α, and let us assume that the probability distribution is such that only L (among all) states have non-zero and equal probability. Then, due to the normalization (1.3), the probability of any of these L states must be equal to $1/L$. According to the definition of the entropy, one gets:

$$S = -\sum_{\alpha}^{L} P_\alpha \log P_\alpha = \log L$$

Therefore, the broader distribution (the larger L), the larger value of the entropy. On the other hand, the more concentrated the distribution function is, the smaller value of the entropy. In the extreme case, when there is only one microscopic state occupied by the system, the entropy is equal to zero. In general, the value of $\exp(S)$ could be interpreted as the averaged number of the states occupied by the system with finite probability.

Now let us consider, what the general form of the probability distribution function must be. According to the basic hypothesis, the average value of the energy of the system is:

$$E \equiv \langle H \rangle = \sum_{\alpha} P_{\alpha} H_{\alpha} \tag{1.5}$$

The interaction of the system with the surrounding world produces the following fundamental effects. First, the averaged value of its energy in the thermal equilibrium is conserved. Second, for some reasons Nature is constructed in such a way that irrespective of the internal structure of the system, the value of the entropy in the equilibrium state tries to attain a maximum (bounded by the condition that the average energy is constant). In a sense, it is natural: random noise makes the system as disordered as possible. Let us now consider, the form of the probability distribution function, which would maximize the entropy. To take into account the two constraints - the conservation of the average energy, Eq.(1.5), and the normalization $\sum_{\alpha} P_{\alpha} = 1$ - one can use the method of the Lagrange multipliers. Therefore, the following expression must be maximized with respect to all possible distributions P_{α}:

$$S_{\beta,\gamma}[P] = -\sum_{\alpha} P_{\alpha} \log(P_{\alpha}) - \beta(\sum_{\alpha} P_{\alpha} H_{\alpha} - E) - \gamma(\sum_{\alpha} P_{\alpha} - 1) \tag{1.6}$$

where β and γ are the Lagrange multipliers. Variation with respect to P_{α} gives:

$$P_{\alpha} = \frac{1}{Z} \exp(-\beta H_{\alpha}) \tag{1.7}$$

where

$$Z = \sum_{\alpha} \exp(-\beta H_{\alpha}) = \exp(\gamma + 1) \tag{1.8}$$

is called the partition function, and the parameter β, which is called the inverse temperature, is defined by the condition:

$$\frac{1}{Z} \sum_{\alpha} H_{\alpha} \exp(-\beta H_{\alpha}) = E \tag{1.9}$$

In practice, however, it is the temperature which is usually taken as an independent parameter, whereas the average energy is obtained as the function of the temperature by Eq.(1.9).

The other fundamental quantity of the statistical mechanics is the free energy defined as follows:

$$F = E - TS \tag{1.10}$$

where $T = 1/\beta$ is the temperature. Using the Eq.(1.7), one can easily derive the following basic relations among the free energy, the partition function, the entropy and the average energy:

$$F = -T \log(Z) \tag{1.11}$$

$$S = \beta^2 \frac{\partial F}{\partial \beta} \tag{1.12}$$

$$E = -\frac{\partial}{\partial \beta} \log(Z) = F + \beta \frac{\partial F}{\partial \beta} \tag{1.13}$$

Note, that according to the definition given by Eq.(1.10), the principle of maximum of entropy is equivalent to that of the minimum of the free energy. One can easily confirm, that taking the free energy (instead of the entropy) as the fundamental quantity which must be minimal with respect to all possible distribution functions, the same form of the probability distribution as given by Eq.(1.7) is obtained.

1.2 The mean-field approximation

In magnetic materials the microscopic state of the system is supposed to be defined by the values of the local spin magnetizations. In many magnetic materials the electrons responsible for the magnetic behavior are localized near the atoms of the crystal lattice, and the force which tends to orient the spins is the (short range) exchange interaction.

The most popular models, which describe this situation qualitatively are called the Ising models. The microscopic variables in these systems are the Ising spins σ_i which by definition can take only two values $+1$ or -1. The traditional form for the microscopic energy (which from now on will be called the Hamiltonian) as the function of all the Ising spins is in the following:

$$H \;=\; -\sum_{<i,j>} J_{ij}\sigma_i\sigma_j \;-\; h\sum_i \sigma_i \tag{1.14}$$

Here the notation $< i, j >$ indicates the summation over all the lattice sites of the nearest neighbors, J_{ij} are the values of the spin-spin interactions, and h is the external magnetic field. If all the J_{ij}'s are equal to a positive constant, then one gets the ferromagnetic Ising model, and if all the J_{ij}'s are equal to a negative constant, then one gets the antiferromagnetic Ising model.

In spite of the apparent simplicity of the Ising model, an exact solution (which means the calculation of the partition function and the correlation functions) has been found only for the one- and the two-dimensional systems in the zero external magnetic field. In all other cases one needs to use approximate methods. One of the simplest methods is called the mean-field approximation. In many cases this method gives the results which are not too far from the correct ones, and very often it makes possible to get some qualitative understanding of what is going on in the system under consideration.

The starting point of the mean-field approximation is the assumption about the structure of the probability distribution function. It is assumed that the distribution function in the equilibrium state can be factorized as the product of the independent distribution functions in the lattice sites:

$$P[\sigma] \;=\; \frac{1}{Z}\exp(-\beta H[\sigma]) \;\simeq\; \prod_i P_i(\sigma_i) \tag{1.15}$$

The normalized site distribution functions are taken in the form:

$$P_i(\sigma_i) \;=\; \frac{1+\phi_i}{2}\delta(\sigma_i - 1) \;+\; \frac{1-\phi_i}{2}\delta(\sigma_i + 1) \tag{1.16}$$

where ϕ_i are the parameters which have to be specified.

The factorization of the distribution function, Eq.(1.15), means that the average of any product of any functions at different sites is also factorizing on the product of the independent averages:

$$< f(\sigma_i)g(\sigma_j) > = < f(\sigma_i) >< g(\sigma_j) > \tag{1.17}$$

where, according to the ansatz (1.15):

$$< f(\sigma_i) > = \; \frac{1+\phi_i}{2}f(1) \;+\; \frac{1-\phi_i}{2}f(-1) \tag{1.18}$$

In particular, for the average site magnetizations, one easily gets:

$$< \sigma_i > = \; \phi_i \tag{1.19}$$

Therefore, the physical meaning of the parameters $\{\phi_i\}$ in the trial distribution function is that they describe the average site spin magnetizations. According to the general principles of the statistical mechanics, these parameters must be such that they would minimize the free energy of the system.

Using Eqs.(1.15) and (1.16) for the entropy and for the average energy, one gets:

$$\begin{aligned} S \;&=\; -< \log(P[\sigma]) > \simeq \; -\textstyle\sum_i < \log(P_i(\sigma_i)) > = \\ &= \; -\textstyle\sum_i [\frac{1+\phi_i}{2}\log(\frac{1+\phi_i}{2}) \;+\; \frac{1-\phi_i}{2}\log(\frac{1-\phi_i}{2}) \end{aligned} \tag{1.20}$$

$$E \;=\; -\frac{1}{2}\sum_{<i,j>} J_{ij}\phi_i\phi_j \;-\; h\sum_i \phi_i \tag{1.21}$$

For the free energy, Eq.(1.10), one obtains:

$$\begin{aligned} F \;&=\; -\tfrac{1}{2}\textstyle\sum_{<i,j>} J_{ij}\phi_i\phi_j \;-\; h\sum_i \phi_i \;+ \\ &+T\textstyle\sum_i[\frac{1+\phi_i}{2}\log(\frac{1+\phi_i}{2}) \;+\; \frac{1-\phi_i}{2}\log(\frac{1-\phi_i}{2})] \end{aligned} \tag{1.22}$$

To be more specific, consider the ferromagnetic system on the D-dimensional cubic lattice. In this case all the spin-spin couplings are equal to some positive constant: $J_{ij} = \frac{1}{2D}J > 0$, (the factor $\frac{1}{2D}$ is inserted just for convenience) and each site has $2D$ nearest neighbors. Since the system is homogeneous, it is natural to expect that all the ϕ_i's must be equal to some constant ϕ. Then, for the free energy (1.22) one gets:

$$\begin{aligned} \tfrac{F}{V} \equiv f(\phi) \;&=\; -\tfrac{1}{2}J\phi^2 \;-\; h\phi \;+ \\ &+T[\tfrac{1+\phi}{2}\log(\tfrac{1+\phi}{2}) \;+\; \tfrac{1-\phi}{2}\log(\tfrac{1-\phi}{2})] \end{aligned} \tag{1.23}$$

where V is the volume of the system and f is the density of the free energy.

The necessary condition for the minimum of f is

$$\frac{df(\phi)}{d\phi} = 0$$

or:

$$-J\phi \;-\; h \;+T\,\mathrm{arctanh}(\phi) \;=\; 0 \tag{1.24}$$

The resulting equation, which defines the order parameter ϕ is:

$$\phi \;=\; \tanh[\beta(J\phi + h)] \tag{1.25}$$

Note, that the minimum of the free energy is conditioned by $\frac{d^2f}{d\phi^2} \;>\; 0$. Using Eq.(1.24), this condition can be reduced to

$$\frac{1}{1-\phi^2} \;>\; \beta J \tag{1.26}$$

Consider first the case of a zero external magnetic field ($h = 0$). One can easily see that if $T > T_c = J$, the only solution of the Eq.(1.25): $\phi = \tanh(\beta J\phi)$ is $\phi = 0$, and this solution satisfies the condition (1.26). Therefore, at all temperatures higher than T_c the minimum of the free energy is achieved in the state in which all the site spin magnetizations are zeros.

However, if $T < T_c$, then in addition to the solution $\phi = 0$ Eq.(1.25) (with $h = 0$) has two non-trivial solutions $\phi = \pm\phi(T) \neq 0$. One can easily check that in this temperature region the solution $\phi = 0$ becomes maximum and not the minimum of the free energy, while the true minima are achieved at $\phi = \pm\phi(T)$. Therefore, in the low temperature region $T < T_c$ the free energy has two minima, which are characterized by non-zero site magnetizations with opposite signs.

Near T_c the magnetization $\phi(T)$ is small. In this case the expansion in powers of ϕ in Eq.(1.25) can be made. In the leading order in $\tau \equiv (T/T_c - 1)$, $|\tau| \ll 1$ one gets:

$$\phi(T) \;=\; const\;|\tau|^{1/2}, \qquad (\tau < 0) \tag{1.27}$$

Thus, as $T \to T_c$ from below, $\phi(T) \to 0$. The expansion of the free energy Eq.(1.23) as the function of small value of ϕ yields:

$$f(\phi) \;=\; \frac{1}{2}\tau\phi^2 \;+\; \frac{1}{4}g\phi^4 \;- h\phi \tag{1.28}$$

where $g = T/3$ and for simplicity we have taken $J = 1$. The qualitative shapes of $f(\phi)$ at $T > T_c$ ($\tau > 0$) and at $T < T_c$ ($\tau < 0$) are shown in Fig.1. Note, that since the total free energy F is proportional to

the volume of the system, the value of the free energy barrier separating the states with $\phi = \pm\phi(T)$ at $T < T_c$ is also proportional to the volume of the system. Therefore, in the *thermodynamic limit* $V \to \infty$ (which corresponds to the consideration of the *macroscopic* systems) the barrier separating the two states is getting infinite.

The simple considerations described above demonstrate on a qualitative level the fundamental phenomenon called spontaneous symmetry breaking. At the temperature $T = T_c$ the phase transition of the second order occurs, such that in the low temperature region $T < T_c$ the symmetry with respect to the global change of the signs of the spins is broken, and the *two* (instead of one) ground states appear. These two states differ by the sign of the average spin magnetization, and they are separated by the macroscopic barrier of the free energy.

1.3 Quenched Disorder, Selfaveraging and the Replica Method

In this lecture course we will consider the thermodynamical properties of various spin systems which are characterized by the presence of some kind of a quenched disorder in the spin-spin interactions. In realistic magnetic materials such disorder can exists, e.g. due to the oscillating nature of the exchange spin-spin interactions combined with the randomness in the positions of the interacting spins (such as in metallic spin-glass alloys $AgMn$), or due to defects in the lattice structure, or because of the presence of impurities, etc.

Since we will be mostly interested in the qualitative effects produced by the quenched disorder, the details of the realistic structure of such magnetic systems will be left aside. Here we will be concentrated on the extremely simplified *model* description of the disordered spin systems.

In what follows we will consider two essentially different types of the disordered magnets. First, we will study the thermodynamic properties of spin systems in which the disorder is strong. The term "strong disorder" refers to the situation when the disorder appears to be the dominant factor for the ground state properties of the system, so that it dramatically changes the low-temperature properties of the magnetic system as compared to the usual ferromagnetic phase. This type of systems, usually called the *spin-glasses*, will be considered in the first part of the course.

In the second part of the course we will consider the properties of weakly disordered magnets. This is the case when the disorder does not produce notable effects for the ground state properties. It will be shown however, that in certain cases even small disorder can produce dramatic effects for the critical properties of the system in a close vicinity of the phase transition point.

The main problem in dealing with disordered systems is that the disorder in their interaction parameters is *quenched*. Formally, all the results one may hope to get for the observable quantities for a given concrete system, must depend on the concrete interaction matrix J_{ij}, i.e. the result would be defined by a macroscopic number of random parameters. Apparently, the results of this type are impossible to calculate, and moreover, they are useless. Intuitively it is clear, however, that the quantities which are called the observables should depend on some general averaged characteristics of the random interactions. This brings us to the concept of the *selfaveraging*.

Traditional way of speculations, why the selfaveraging phenomenon should be expected to take place, is as follows. The free energy of the system is known to be proportional to the volume V of the system. Therefore, in the thermodynamic limit $V \to \infty$ the main contribution to the free energy must come from the volume, and not from the boundary, which usually produces the effects of the next orders in the small parameter $1/V$. Any macroscopic system could be divided into macroscopic number of macroscopic subsystems. Then the total free energy of the system would consist of the sum of the free energies of the subsystems, plus the contribution which comes from the interactions of the subsystems, at their boundaries. If all the interactions in the system are short range (which takes place in any realistic system), then the contributions from the mutual interactions of the subsystems are just the boundary effects which vanish in the thermodynamic limit. Therefore, the total free energy could be represented as a sum of the macroscopic number of terms. Each of these terms would be a random quenched quantity since it contains, as the parameters, the elements of the random spin-spin interaction matrix. In accordance with the law of large numbers, the sum of many random quantities can be represented as their average value, obtained from their statistical distribution, times their number (all this is true, of course, only under certain requirements on the characteristics of the

statistical distribution). Therefore, the total free energy of a macroscopic system must be selfaveraging over the *realizations* of the random interactions in accordance with their statistical distribution.

The free energy is known to be given by the logarithm of the partition function. Thus, in order to calculate the observable thermodynamics one has to average the logarithm of the partition function over the given distribution of random J_{ij}'s after the calculation of the partition function itself. To perform such a program the following technical trick, which is called the replica method, is used.

Formally, the replicas are introduced as follows. In order to obtain the physical (selfaveraging) free energy of the quenched random system we have to average the logarithm of the partition function:

$$F \equiv \overline{F_J} = -\frac{1}{\beta}\overline{\ln(Z_J)} \tag{1.29}$$

where $\overline{(...)}$ denotes the averaging over random interactions $\{J_{ij}\}$ with a given distribution function $P[J]$:

$$\overline{(...)} \equiv (\prod_{<i,j>}\int dJ_{ij})P[J](...) \tag{1.30}$$

and the partition function is

$$Z_J = \sum_{\sigma}\exp\{-\beta H[J,\sigma]\} \tag{1.31}$$

To perform this procedure of the averaging, the following trick is invented. Let us consider the *integer* power n of the partition function (1.31). This quantity is the partition function of the n non-interacting *identical* replicas of the original system (i.e. having identical fixed spin-spin couplings J_{ij}):

$$Z_J^n = (\prod_{a=1}^{n}\sum_{\sigma^a})\exp\{-\beta\sum_{a=1}^{n}H[J,\sigma_a]\} \tag{1.32}$$

Here the subscript a labels the replicas. Let us introduce the quantity:

$$F_n = -\frac{1}{\beta n}\ln(Z_n) \tag{1.33}$$

where

$$Z_n \equiv \overline{Z_J^n} \tag{1.34}$$

Now, if a *formal* limit $n \to 0$ would be taken in the expression (1.33), then the original expression for the physical free energy (1.29) will be recovered:

$$\begin{aligned}&\lim_{n\to 0} F_n = -\lim_{n\to 0}\tfrac{1}{\beta n}\ln(Z_n) = -\lim_{n\to 0}\tfrac{1}{\beta n}\ln[\overline{\exp\{n\ln Z_J\}}] = \\ &-\tfrac{1}{\beta}\overline{\ln Z_J} = F\end{aligned} \tag{1.35}$$

Thus, the scheme of the replica method can be described in the following steps. First, the quantity F_n for the integer n must be calculated. Second, the analytic continuation of the obtained function of the parameter n should be made for an arbitrary non-integer n. Finally, the limit $n \to 0$ has to be taken.

Although this procedure may look rather doubtful at first, actually it is not so creasy. First, if the free energy appears to be an analytic function of the temperature and the other parameters (so that it can be represented as the series in powers of β), then the replica method can be easily proved to be correct in a strict sense. Second, in all cases, when the calculations can be performed by some other method, the results of the replica method are confirmed.

One could also introduce replicas in the other way [2],[20],[21]. Let us consider a general spin system described by some Hamiltonian $H[J;\sigma]$, which depends on the spin variables $\{\sigma_i\}$ and the spin-spin interactions J_{ij} (the concrete form of the Hamiltonian is irrelevant). If the interactions J_{ij} are quenched, the free energy of the system would depend on the concrete realization of the J_{ij}'s:

$$F[J] \;=\; -\frac{1}{\beta}\log(Z_J) \tag{1.36}$$

Now, let us assume that the spin-spin interactions are *partially annealed* (i.e. not perfectly quenched), so that they can also change their values, but the characteristic time scale of their changes is much larger than the time scale at which the spin degrees of freedom reach the thermal equilibrium. In this case the free energy given by (1.36) would still make sense, and it would become the energy function (the Hamiltonian) for the degrees of freedom of J_{ij}'s.

Besides, the space in which the interactions J_{ij} take their values should be specified separately. The interactions J_{ij}'s could be discrete variables taking values $\pm J_0$, or they could be the continuous variables taking values in some restricted interval, or they could be something else. In the quenched case this space of J_{ij} values is defined by a statistical distribution function $P[J]$. In the case of the partial annealing this function $P[J]$ has a meaning of the internal potential for the interactions J_{ij}, which restricts the space of their values.

Let us now assume, that the spin and the interaction degrees of freedom *are not thermally equilibrated*, so that the degrees of freedom of the interactions have their own temperature T', which is different from the temperature T of the spin degrees of freedom. In this case for the total partition function of the system one gets:

$$\begin{aligned} \mathcal{Z} \;&=\; \int DJP[J]\exp(-\beta' F[J]) \;= \\ &\int DJP[J]\exp(\tfrac{\beta'}{\beta}\log Z_J) \;= \\ &\int DJP[J](Z_J)^n \equiv \overline{(Z_J)^n} \end{aligned} \tag{1.37}$$

where $n = T/T'$. Correspondingly, the total free energy of the system would be:

$$\mathcal{F} \;=\; -T'\log[\overline{(Z[J])^n}] \tag{1.38}$$

In this way we have arrived to the replica formalism again, in which the "number of replicas" $n = T/T'$ appears to be the *finite* parameter.

To obtain the physical (selfaveraging) free energy in the case of the quenched random J_{ij}'s one takes the limit $n \to 0$. From the point of view of the partial annealing, this situation corresponds to the limit of the infinite temperature T' in the system of J_{ij}'s. This is natural in a sense that in this case the thermodynamics of the spin degrees of freedom produces no effect on the distribution of the spin-spin interactions.

In the case when the spin and the interaction degrees of freedom are thermally equilibrated, $T' = T$ ($n = 1$), we arrive at the trivial case of the purely annealed disorder, irrespective of the difference between the characteristic time scales of the J_{ij} interactions and the spins. This is also natural because the thermodynamic description formally corresponds to the infinite times, and the characteristic time scales of the dynamics of the internal degrees of freedom become irrelevant. If $n \neq 0$ and $n \neq 1$, one gets the situation, which could be called the partial annealing, and which is the intermediate case between quenched disorder and annealed disorder.

Part I. SPIN-GLASS SYSTEMS

2 Physics of the Spin Glass State

Before starting doing detailed calculations, first it would be useful to get qualitative understanding of the general physical phenomena taking place in statistical mechanics of spin systems with strong quenched disorder. Therefore, in this Chapter we will discuss the problem of spin glass state only in simple qualitative terms.

2.1 Frustrations

There are exists quite a few statistical models of spin glasses. Here we will be concentrated on one of the simplest models which can be formulated in terms of the classical Ising spins, described by the following Hamiltonian:

$$H = -\frac{1}{2}\sum_{i\neq j}^{N} J_{ij}\sigma_i\sigma_j \tag{2.1}$$

This system consists of N Ising spins $\{\sigma_i\}$ $(i = 1, 2, ..., N)$, taking values ± 1 which are placed in the vertices of some lattice. The spin-spin interactions J_{ij} are random in their values and signs. The properties of such system are defined by the statistical distribution function $P[J_{ij}]$ of the spin-spin interactions. For the moment, however, the concrete form of this distribution will not be important. The motivation for the Hamiltonian (2.1) from the point of view of realistic spin-glass systems is well described in the review [3].

The crucial phenomena revealed by strong quenched disorder, which makes such type of systems so hard to study, is in the following. Consider the system of three interacting spins (Fig.2). Let us assume for simplicity, that the interactions among them can be different only in their signs being equal in the absolute value. Then for the ground state of such system we can find two essentially different situations.

If all three interactions J_{12}, J_{23} and J_{13} are positive, or two of them are negative while the third one is positive, then the ground state of this three spin system is unique (except for the global change of signs of all the spins) (Fig.2a). This is the case when the product of the interactions along the triangle is positive.

However, if the product of the interactions along the triangle is negative (one of the interactions is negative, or all three interactions are negative, Fig.2b), then the ground state of such a system is degenerate. One can easily check, going from spin to spin along the triangle, that in this case the orientation ("plus" or "minus") of one of the spins remains "unsatisfied" with respect to the interactions with its neighbors.

One can also easily check that similar phenomenon takes place in any closed spin chain of arbitrary length, provided that the product of the spin-spin interactions along the chain is negative. This phenomenon is called *frustration*[1] [4].

One can easily see that not *any* disorder induces frustrations. On the other hand, it is the frustrations, which describe the relevant part of the disorder, and which essentially effect the ground state properties of the system. In other words, if the disorder does not produce frustrations, it can be considered as being irrelevant. In some cases an irrelevant disorder can be just removed by a proper redefinition of the spin variables of the system. A simple example of this situation is illustrated by the so-called Mattice magnet. This is also formally disordered spin system, which is described by the Hamiltonian (2.1), where the spin-spin interactions are defined as follows: $J_{ij} = \xi_i\xi_j$, and the quenched ξ_i's are taking values ± 1 with equal probability. In such system the interactions J_{ij} are also random in signs, although one can easily check that with such concrete definition of the random interactions no

[1] This term is quite adequate in its literal meaning, since the triangle discussed above might as well be interpreted as the famous love triangle. Besides, the existence of frustrations in spin glasses breaks any hope for finding a simple solution of the problem.

frustrations appear in the system. Moreover, after simple redefinition of the spin variables: $\sigma_i \to \sigma_i \xi_i$, an ordinary ferromagnetic Ising model will be recovered. Thus, this type of disorder (it is called the Mattice disorder) is actually fictitious for the thermodynamic properties of the system.

It is crucial that the "true" disorder with frustrations can not be removed by any transformation of the spin variables. Since in a macroscopic spin system, in general, one can draw a lot of different frustrated closed spin chains, the total number of frustrations must be also macroscopically large. This, in turn, would either result in a tremendous degeneracy of the ground state, or, in general, it could produce a lot of low-lying states with the energies very close to the ground state. In particular, in the Ising spin glass described by the Hamiltonian (2.1) the total number of such states is expected to be of the order of $\exp(\lambda N)$ (where $\lambda < \ln 2$ is a numerical factor), while the total amount of states in this system is equal to $2^N = \exp[(\log 2)N]$.

2.2 Ergodicity breaking

Formally, according to the general selfaveraging arguments (Section 1.3), to derive the observable thermodynamics of a disordered spin system one has to find a way for averaging the logarithm of the partition function over random parameters J_{ij} simultaneously with the calculation of the partition function itself. It is clear that this problem is not easy, but nevertheless, at the level of such type of very general speculations it looks as if it is just a technical problem (well, presumably very hard one), and not more than that. Actually, for spin-glass type of systems this is not just the *technical* problem. To realize this, let us consider again a few general points of the statistical mechanics.

Everything would be rather simple if the free energy in the thermodynamic limit would be an analytic function of the temperature and the other parameters. Actually for most of non-trivial systems which are of interest in the statistical mechanics this is not so. Very often due to spontaneous breaking of some kind of a symmetry in the thermodynamic limit there exist a *phase transition* in the system under consideration, and this makes the free energy to be a *non-analytic* function of the parameters involved.

Let us consider again the ordinary ferromagnetic Ising model (Chapter 1) which in very simple terms illustrates the physical consequences of this phenomenon. Since the Hamiltonian of this system is invariant with respect to the global change of the signs of all the spins, any thermodynamic quantity which is odd in spins must be identically equal to zero. In particular, this must be true for the quantity which describes the global magnetization of the system. If the volume N of the system is *finite* these arguments are indeed perfectly correct. However, in the thermodynamic limit $N \to \infty$ we are facing rather non-trivial situation. According to simple calculations performed in Section 1.2 the free energy as function of the global magnetization acquires the double-well shape (Fig.1) at low temperatures. The value of the energy barrier separating the two ground states is proportional to the volume of the system, and it is getting infinite in the limit $N \to \infty$. In other words, at temperatures below T_c the space of all microscopic states of the system is getting to be divided into two equal valleys separated by the infinite barrier. On the other hand, according to the fundamental ergodic hypothesis of the statistical mechanics (Section 1.1) it is assumed that in the limit of infinite observation time the system (following its internal dynamics) visits all its microscopic states many times, and it is this assumption which makes possible to apply the statistical mechanical approach: for the calculation of the averaged quantities we use averaging over the ensemble of states with the corresponding probability distribution instead of that over the time. In the situation under consideration, when the thermodynamic limit $N \to \infty$ is taken *before* the observation time goes to infinity (it is this order of limits which corresponds to the adequate statistical mechanical description of a *macroscopic* system) the above ergodic assumption simply does not work. Whatever the (reasonable) internal dynamics of the system is, it could never makes possible to jump over the infinite energy barrier separating the two valleys of the space of states. Thus, in the *observable* thermodynamics only half of the states contribute, (these are the states which are on one side from the barrier), and that is why in the observable thermodynamics the global magnetization of the system appears to be non-zero.

In the terminology of the statistical mechanics this phenomenon is called the *ergodicity breaking*, and it manifest itself as the *spontaneous symmetry breaking*: below T_c the observable thermodynamics is getting non-symmetric with respect to the global change of signs of all the spins. As a consequence, in the calculations of the partition function below T_c one has to take into account not all, but only one half of all the microscopic states of the system (the states which belong to one valley).

The above example of the ferromagnetic system is very simple because here one can easily guess right away what kind of the symmetry could be broken at low temperatures. In spin glasses the spontaneous symmetry breaking also takes place. However, unlike the ferromagnetic system, here it is much more difficult to guess which one. The main problem is that the symmetry which might be broken in a given sample can depend on the quenched disorder parameters involved. In this situation the calculation of the observable free energy is getting extremely difficult problem because, unlike naive plain summation over all the microscopic states, one must take into account only the states belonging to one of the many valleys, while the structure of these valleys (which, in general, can appear to be non-equivalent) depends on a concrete realization of the random disorder parameters.

2.3 Continuous sequence of the phase transitions

Of course, the existence of many local minima states in the frustrated spin system does not automatically means that at low temperatures some of these states create their valleys separated by the infinite barriers of the free energy. Due to thermal fluctuations (which are usually rather strong in the low-dimensional systems) the energy barriers could effectively "melt", and in this case the only ground state of the free energy could appear to be the one with the zero local spin magnetization. Then there will be no spontaneous symmetry breaking, and at any finite temperature the system will be in the "symmetric" paramagnetic state. Of course, from the point of view of the anomalously slow dynamic relaxation properties this state can be essentially different from the usual high-temperature paramagnetic state, but this problem would lead us well beyond the scope of pure statistical mechanics.

It could also happen that due to some symmetry properties, thermal fluctuations etc., the global minimum of the free energy of a given sample is achieved at low temperatures at some unique non-trivial spin configuration (of course, in this case the "counterpart" spin state which differ by the global change of the signs of the spins must also be the ground state). It would mean that at low enough temperatures (below certain phase transition temperature T_c) the system must "freeze" in this unique random spin state, which will be characterized by the non-zero values of the thermally averaged local spin magnetizations at each site $\langle \sigma_i \rangle$. Since this ground state is random, the values of the local magnetizations $\langle \sigma_i \rangle$ will fluctuate in their values and signs from site to site, so that the usual ferromagnetic order parameter, which describes the global magnetization of the system: $m = \frac{1}{N} \sum_i \langle \sigma_i \rangle$ must be zero (in the infinite volume limit). However, this state can be characterized by the other order parameter (usually called the Edwards-Anderson order parameter [5]):

$$q \;=\; \frac{1}{N} \sum_i \langle \sigma_i \rangle^2 \;\neq 0 \tag{2.2}$$

The properties of the systems of this type is studied in details in the papers by Fisher and Huse [6], and we will not consider them here.

In the subsequent Chapters we will concentrate on a qualitatively different situation, which arises when there exist macroscopically large number of spin states in which the system could get "frozen" at low temperatures. Moreover, unlike "ordinary" statistical mechanical systems, according to the mean-field theory of spin glasses the spontaneous symmetry breaking in the spin-glass state takes place not just at certain T_c, but it occurs at *any* temperature below T_c. In other words, below T_c a *continuous sequence* of the phase transitions takes place, and correspondingly the free energy appears to be non-analytic at any temperature below T_c.

In general qualitative terms this phenomenon can be described as follows. Just below certain critical temperature T_c the space of spin states is divided into *many* valleys (their number diverges in the thermodynamic limit), separated by infinite barriers of the free energy. At the temperature $T = T_c - \delta T$ each valley is characterized by the non-zero values of the average local spin magnetizations $\langle \sigma_i \rangle_{(\alpha)}$ (which, of course, fluctuate in a sign and magnitude from site to site). Here $\langle ... \rangle_\alpha$ denotes the thermal average inside a particular valley number α. The order parameter, which would describe the degree of freezing of the system inside the valleys could be defined as follows:

$$q(T) \;=\; \frac{1}{N} \sum_i [\langle \sigma_i \rangle_{(\alpha)}]^2 \tag{2.3}$$

According to the mean-field theory of spin-glasses the value of q depends only on the temperature, and it appears to be the same for all the valleys. At $T \to T_c, \quad q(T) \to 0$.

At further decrease of the temperature new phase transitions of ergodicity breaking takes place, so that each valley is divided into many new smaller ones separated by infinite barriers of the free energy (Fig.3). The state of the system in all new valleys can be again characterized by the order parameter (2.3), and its value is growing while the temperature is decreasing.

As the temperature goes down to zero, this process of fragmentation of the space of states into smaller and smaller valleys goes on *continuously*. In a sense, it means that at *any* temperature below T_c the system is in the critical state.

To what extent this situation is realistic from the experimental point of view remains open, although the series of recent experiments (which will be discussed in Chapter 9) gives strong indication in favour of it. In any case, this new type of physics is very interesting in itself, and it is worthing to be studied.

2.4 Order parameter

It is clear that the order parameter (2.3) defined for one valley only, does not contain any information about the other valleys, and it does not tell anything, about the structure of space the ground states as a whole. Let us try to construct the other physical order parameter, which would describe this structure as fully as possible.

Consider the following series of imaginary experiments. Let us take an arbitrary disordered spin state, and then at a given temperature T below T_c let the system relax to the thermal equilibrium. For each experiment a new starting random spin state should be taken. Then each experiment we will be characterized by some equilibrium values of the average local spin magnetizations $\langle\sigma_i\rangle_{(\alpha)}$, where α denotes the number of the experiment. Since there exists macroscopically large number of valleys in the phase space in which the system could get "trapped" these site magnetizations, in general, could be different for different experiments.

Let us assume that we have performed infinite number of such experiments. Then, we can introduce the quantity, which would describe to what extent the states which have been obtained in different experiments are close to each other:

$$q_{\alpha\beta} \;=\; \frac{1}{N}\sum_{i}^{N}\langle\sigma_i\rangle_{(\alpha)}\langle\sigma_i\rangle_{(\beta)} \tag{2.4}$$

It is clear that $|q_{\alpha\beta}| \leq 1$, and the maximum value of $q_{\alpha\beta}$ is achieved when the two states in the experiments α and β coincide (in this case the overlap (2.4) coincides with that of (2.3), which has been introduced for one valley only). It is also clear, that the less correlated the two different states are, the smaller value of the overlap (2.4) they have. If the two states are not correlated at all, then their overlap (in the thermodynamic limit) is equal to zero. In this sense the overlap $q_{\alpha\beta}$ defines a kind of a metrics in the space of states (the quantity $q_{\alpha\beta}^{-1}$ could be conditionally called the "distance" in the space of states).

To describe the statistics of the overlaps in the space of these states one can introduce the following probability distribution function:

$$P(q) \;=\; \sum_{\alpha\beta}\delta(q_{\alpha\beta} - q) \tag{2.5}$$

It appears that it is in terms of this distribution function $P(q)$ the spin glass state looks essentially different from any other "ordinary" thermodynamic state.

Possible types of the functions $P(q)$ is shown in Fig.4. The paramagnetic phase is characterized by the only global minimum of the free energy, in which all the site magnetizations are equal to zero. Therefore the distribution functions $P(q)$ in this phase is the δ-function at $q = 0$ (Fig.4a). In the ferromagnetic phase there are exist two minima of the free energy with the site magnetizations $\pm m$. Thus, the distribution function $P(q)$ in this phase must contain two δ-peaks at $q = \pm m^2$ (Fig.4b). It is clear that in the case of the "fake" spin glass phase in which there exist only two global minima disordered spin states (the states which differ by the global reversal of the local spin magnetizations) the distribution function $P(q)$ must look the same as in the ferromagnetic state.

According to the mean-field theory of spin-glasses, which will be considered in the subsequent Chapters, the distribution function $P(q)$ in the "true" spin glass phase looks essentially different (Fig.4c). Here, between the two δ-peaks at $q = \pm q_{max}(T)$ there is a continuous curve. The value of $q_{max}(T)$ is equal to the maximum possible overlap of the two ground states which is the "selfoverlap" (2.3). Since the number of the valleys in the system is macroscopically large and their selfoverlaps are all equal, the function $P(q)$ has two δ-peaks at $q = \pm q_{max}(T)$. The existence of the continuous curve in the interval $(0, \pm q_{max}(T))$ is the direct consequence of the "origin" of the spin states involved: since they appear as the result of a continuous process of fragmentation of the valleys into the smaller and smaller ones, the states which form such type of the *hierarchy* are getting necessary correlated.

Thus, it is the distribution function $P(q)$ which can be considered as the proper physical order parameter, adequately describing the peculiarities of the spin-glass phase. Although the procedure of its definition described above looks somewhat artificial, later it will be shown that the distribution function $P(q)$ can be defined as the thermodynamical quantity, and moreover, in terms of the mean-field theory of spin-glasses it can be calculated explicitly.

2.5 Ultrametricity

According to the qualitative picture described above, the spin-glass states are organized in a kind of a hierarchical structure (Fig.3). It can be proved that this rather sophisticated space of states could be described in terms of the well defined thermodynamical quantities.

In the previous Section we have introduced the distribution function $P(q)$, which gives the probability to find two spin glass states having the overlap equal to q. Now let us introduce somewhat more complicated distribution function $P(q_1, q_2, q_3)$ which gives the probability for arbitrary *three* spin glass states to have their overlaps to be equal to q_1, q_2 and q_3:

$$P(q_1, q_2, q_3) = \sum_{\alpha\beta\gamma} \delta(q_{\alpha\beta} - q_1)\delta(q_{\alpha\gamma} - q_2)\delta(q_{\beta\gamma} - q_3) \tag{2.6}$$

In terms of the mean-field theory for the model of the spin glasses with the long range interactions this function can be calculated explicitly (Chapter 5). It can be shown that the function $P(q_1, q_2, q_3)$ is not equal to zero only if at least two of its three overlaps are equal to each other and their value is not larger than the third one. In other words, the function $P(q_1, q_2, q_3)$ is non-zero only in the following three cases: $q_1 = q_2 \leq q_3$; $\quad q_1 = q_3 \leq q_2$; $\quad q_3 = q_2 \leq q_1$. In all other cases the function $P(q_1, q_2, q_3)$ is identically equal to zero. It means that in the space of spin glass states there exist no triangles with all three sides being different. The spaces having the above metric property are called *ultrametric*. The ultrametricity from the point of view of physics (in mathematics the ultrametric structures was known since the end of the last century) is described in details in the review [7].

The most simple illustration of the ultrametric structure can be made in terms of the hierarchical tree (Fig.5). Here the space of the spin glass states is identified with the set of the endpoints of the tree. The metric in this space is defined in such a way, that the overlap (the distance) between any two states depends only on the number of generations to their closest "ancestor" on the tree (as the number of the generations increases, the value of the overlap decreases). One can easily check that the space with such metrics is ultrametric.

It the mean-field theory of spin-glasses such illustrative tree of states actually describes the hierarchical fragmentation of the space of the spin-glass states into the valleys, as it has been described above (Fig.3). If for the vertical axis in the Fig.5 we assign the (discrete) value of the paired overlaps q, then the set of the spin glass states at any given temperature $T < T_c$ can be obtained at the crossection of the tree at the level $q = q_{max}(T)$. After decreasing the temperature to a new value $T' < T$, each of the states at the level $q_{max}(T)$ gives birth to a numerous "descendants", which are the endpoints of the tree at the new level $q_{max}(T') > q_{max}(T)$. Correspondingly, after increasing the temperature to a higher value $T'' > T$, all the states having their common ancestors at the level $q_{max}(T'') < q_{max}(T)$ merge together into one state. As $T \to T_c$, $q_{max}(T) \to 0$, which is the level of the (paramagnetic) "grandancestor" of all the spin glass states.

Since the function $q_{max}(T)$ is determined by the temperature, it means that it is the temperature which defined the level of the tree at which the "horizontal" crossection should be made, and this, in turn, reveal all the spin glass states at this temperature. All the states which are below this level

are "indistinguishable", while all the states which are above this level form the "evolution history" of the spin glass states at a given temperature. In this sense the temperature defines the elementary ("ultraviolet") *scale* in the space of the spin glass states. This creates a kind of scaling in the spin glass phase: by changing the temperature one just changes the scale in the space of the spin-glass states.

3 Mean Field Theory of Spin Glasses

3.1 Infinite range interaction model

The Sherrington and Kirkpatric (SK) model of spin glasses [8] is defined by the usual Ising spins Hamiltonian:

$$H \;=\; -\sum_{i<j}^{N} J_{ij}\sigma_i\sigma_j \tag{3.1}$$

where the spin-spin interactions J_{ij} are random quenched variables which are described by the symmetric Gaussian distribution *independent* for any pair of sites (i,j):

$$P[J_{ij}] \;=\; \prod_{i<j}[\sqrt{\frac{N}{2\pi}}\exp\{-\frac{N}{2}J_{ij}^2\}] \tag{3.2}$$

According to the above definition, each spin interacts with all the other spin of the system. For that reason the space structure (dimensionality, type of the lattice, etc.) of this model is irrelevant for its properties. The space here is just the set of N sites in which the Ising spins are placed, and all these spins, in a sense, could be considered as the nearest neighbors. In the thermodynamic limit ($N\to\infty$) such structure can be interpreted as the infinite dimensional lattice, and it is this property which makes the mean-field approach to be exact.

According to the probability distribution (3.2) one gets:

$$\overline{J_{ij}} \;=\; 0\,; \qquad \overline{J_{ij}^2} \;=\; \frac{1}{N} \tag{3.3}$$

where $\overline{(...)}$ denotes the averaging over random J_{ij}'s:

$$\overline{(...)} \equiv \int D\mathbf{J}P[\mathbf{J}](...) \;=\; \prod_{i<j}[\sqrt{\frac{N}{2\pi}}\int_{-\infty}^{+\infty} dJ_{ij}\exp\{-\frac{N}{2}J_{ij}^2\}](...) \tag{3.4}$$

One could easily check that due to the chosen normalization of the order of $1/N$ for the average square values of the couplings J_{ij}, the average energy of the system appears to of the order of N, as it should be for an adequately defined physical system.

It is clear, of course, that microscopic structure of the model defined above is completely unphysical. Nevertheless, this model has two big advantages: first, it is exactly solvable, and second, its solution appears to be quite non-trivial. Moreover, on a qualitative level the physical interpretation of this solution, hopefully, could be also generalized for "normal" random physical systems. If it would be discovered (e.g. in experiments) that real spin glasses demonstrate the physical properties predicted due to the solution of the SK model, then, in a sense, it is not so important, what was the original artificial system, which has initiated the true result.

3.2 Replica symmetric solution

To calculate the replica free energy F_n, eq.(1.33), according to the eqs. (1.32)-(1.34) one has to calculate the annealed average of the n-th power of the partition function:

$$Z_n \;=\; \sum_{\sigma_i^a}\int DJ_{ij}\exp\{\beta\sum_{a=1}^{n}\sum_{i<j}^{N}J_{ij}\sigma_i^a\sigma_j^a - \frac{N}{2}\sum_{i<j}J_{ij}^2\} \tag{3.5}$$

(here and in what follows irrelevant pre-exponential factors are omitted). The integration over J_{ij}'s gives:

$$Z_n \;=\; \sum_{\sigma_i^a} \exp\{\frac{\beta^2}{2N} \sum_{i<j}^{N} (\sum_{a=1}^{n} \sigma_i^a \sigma_j^a)^2\} \tag{3.6}$$

or:

$$Z_n \;=\; \sum_{\sigma_i^a} \exp\{\frac{1}{4}\beta^2 N n + \frac{1}{2}\beta^2 N \sum_{a<b}^{n} (\frac{1}{N} \sum_{i}^{N} \sigma_i^a \sigma_i^b)^2\} \tag{3.7}$$

The summation over the sites in the above equation can be linearized by introducing the replica matrix Q_{ab}:

$$Z_n \;=\; \prod_{a<b}^{n} (\int dQ_{ab}) \sum_{\sigma_i^a} \exp\{\frac{1}{4}\beta^2 N n - \frac{1}{2}\beta^2 N \sum_{a<b}^{n} Q_{ab}^2 + \beta^2 \sum_{a<b}^{n} \sum_{i}^{N} Q_{ab} \sigma_i^a \sigma_i^b\} \tag{3.8}$$

The replica variables Q_{ab} have clear physical interpretation. According to the above equation, the equilibrium values of the matrix elements Q_{ab} are defined by the equations $\delta Z_n / \delta Q_{ab} = 0$, which give:

$$Q_{ab} \;=\; \frac{1}{N} \sum_{i}^{N} \langle \sigma_i^a \sigma_i^b \rangle \tag{3.9}$$

Since the expression in the exponent of the eq.(3.8) is linear in the spatial summation, the total partition function can be factorized into the independent site partition functions:

$$Z_n \;=\; \prod_{a<b}^{n} (\int dQ_{ab}) \exp\{\frac{1}{4}\beta^2 N n - \frac{1}{2}\beta^2 N \sum_{a<b}^{n} Q_{ab}^2\} \prod_{i}^{N} [\sum_{\sigma_i^a} \exp\{\beta^2 \sum_{a<b}^{n} Q_{ab} \sigma_i^a \sigma_i^b\}] \tag{3.10}$$

or

$$Z_n \;=\; \prod_{a<b}^{n} (\int dQ_{ab}) \exp\{\frac{1}{4}\beta^2 N n - \frac{1}{2}\beta^2 N \sum_{a<b}^{n} Q_{ab}^2 + N \log[\sum_{\sigma_a} \exp(\beta^2 \sum_{a<b}^{n} Q_{ab} \sigma_a \sigma_b)]\} \tag{3.11}$$

This equation can be represented as follows:

$$Z_n \;=\; \int D\hat{Q} \exp(-\beta n N f_n[\hat{Q}]) \tag{3.12}$$

where

$$f_n[\hat{Q}] = -\frac{1}{4}\beta + \frac{1}{2n}\beta \sum_{a<b}^{n} Q_{ab}^2 - \frac{1}{\beta n} \log[\sum_{\sigma_a} \exp(\beta^2 \sum_{a<b}^{n} Q_{ab} \sigma_a \sigma_b)] \tag{3.13}$$

In the thermodynamic limit the integral for the partition function (3.12) in the leading order in N is given by the saddle point of the function $f[\hat{Q}]$:

$$Z_n \;\simeq\; [det \frac{\delta^2 f}{\delta \hat{Q}^2}]^{(-1/2)} \exp(-\beta n N f[\hat{Q}^*]) \tag{3.14}$$

Here $\hat{Q}^*$ is the matrix corresponding to the minimum of the function f, and it is defined by the saddle-point equation:

$$\frac{\delta f}{\delta Q_{ab}} \;=\; 0 \tag{3.15}$$

According to a general scheme of the replica method, the quantity $f[\hat{Q}^*]$ in the limit $n \to 0$ gives the density of the free energy of the system.

Thus, further strategy should be in the following. First, for an arbitrary matrix $\hat{Q}$ one has to calculate an explicit expression for the replica free energy (3.13). Then one has to find the solution $\hat{Q}^*$ of the saddle-point equations (3.15) and the corresponding value for the replica free energy $f_n[\hat{Q}^*]$. Finally the limit $\lim_{n\to 0} f_n[\hat{Q}^*]$ should be taken. Unfortunately, this systematic program can not be fulfilled, because for an *arbitrary* matrix $\hat{Q}$ the replica free energy (3.13) can not be calculated.

Therefore, the procedure of solving the problem is getting somewhat more intuitive. First, one has to *guess* the correct structure of the solution $\hat{Q}^*$, which would hopefully depend on the limited number of parameters, and which would make possible to calculate the replica free energy (3.13). Then these parameters should be obtained from the saddle-point equations (3.15), and finally the corresponding value of the saddle-point free energy should be calculated. Of course, according to this scheme, one would be able to find the extremum only inside some limited subspace of all matrices $\hat{Q}$. However, if it would be possible to prove that the corresponding Hessian $\delta^2 f/\delta\hat{Q}^2$ at this extremum is positively defined, then it would mean that the true extremum is found. (Of course, this scheme does not guarantee that there exist no others saddle points.)

Since all the replicas in our system are equivalent, one could naively guess that the adequate form of the matrix $\hat{Q}^*$ is such that all its elements are equal:

$$Q_{ab} \;=\; q\,; \qquad \text{for all } a \neq b \tag{3.16}$$

This *ansatz* is called the replica symmetric (RS) approximation.

All the calculations in the RS approximation are very simple. For the replica free energy (3.13) one gets:

$$f(q) = -\frac{1}{4}\beta + \frac{\beta}{4n}n(n-1)q^2 - \frac{1}{\beta n}\log[\sum_{\sigma_a}\exp\{\frac{1}{2}\beta^2(\sum_a^n \sigma_a)^2 q - \frac{1}{2}\beta^2 nq\}] \tag{3.17}$$

In the standard way introducing the Gaussian integration one makes the quadratic term in the exponent to be linear in σ's:

$$f(q) = -\frac{1}{4}\beta + \frac{1}{2}\beta q + \frac{1}{4}(n-1)\beta q^2 - \frac{1}{\beta n}\log[\int_{-\infty}^{+\infty}\frac{dz}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)\sum_{\sigma_a=\pm 1}\exp\{\beta z\sqrt{q}\sum_a^n \sigma_a\}] \tag{3.18}$$

Summing over σ's one gets:

$$f(q) = -\frac{1}{4}\beta + \frac{1}{2}\beta q + \frac{1}{4}(n-1)\beta q^2 - \frac{1}{\beta n}\log\left(\int_{-\infty}^{+\infty}\frac{dz}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)[2\cosh(\beta z\sqrt{q})]^n\right) \tag{3.19}$$

Taking the limit $n \to 0$ one finally obtains:

$$f(q) = -\frac{1}{4}\beta(1-q)^2 - \frac{1}{\beta}\int_{-\infty}^{+\infty}\frac{dz}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)\ln\left(2\cosh(\beta z\sqrt{q})\right) \tag{3.20}$$

Now one can easily derive the corresponding saddle-point equation for the parameter q:

$$q \;=\; \int_{-\infty}^{+\infty}\frac{dz}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)\tanh^2(\beta z\sqrt{q}) \tag{3.21}$$

One can easily check that at $T \geq T_c = 1$ the only solution of this equation is $q = 0$. On the other hand, at $T < T_c$ there exists non-trivial solution $q(T) \neq 0$. In the vicinity of the critical temperature, at $(1-T) \equiv \tau << 1$, this solution can be found explicitly: $q(\tau) \simeq \tau$. It is also easy to check that in the low temperature limit $T \to 0$, $q(T) \to 1$.

According to eqs.(3.16) and (3.9), the obtained solution for $q(T)$ gives us the physical order parameter:

$$q(T) \;=\; \frac{1}{N}\sum_i^N \langle\sigma_i\rangle^2 \tag{3.22}$$

Since $q(T)$ is not equal to zero in the low temperature region, $T < T_c$, the spins of the system must be getting frozen in some random state. Besides, since there exists only one solution for $q(T)$, such a disordered ground state must be unique.

After the result for the free energy is derived, one can easily perform further straightforward calculations to obtain the results for all the observable thermodynamical quantities, such as specific heat, susceptibility, entropy etc. Thus, in terms of the considered replica symmetric ansatz a complete solution of the problem can be easily obtained.

All that would be very nice, if it would be correct. Unfortunately, it is not. One of the simplest way to see that there is something fundamentally wrong in the obtained solution is to calculate the entropy. One can easily check that at sufficiently low temperatures the entropy is getting negative! (At $T = 0$ the entropy $S = -\frac{1}{2\pi} \simeq -0.17$). Moreover, the calculations of the Hessian $\delta^2 f/\delta\hat{Q}^2$ for the obtained RS solution (see Appendix) demonstrate that this solution appears to be *unstable* ($det(\delta^2 f/\delta\hat{Q}^2) < 0$) in all the low temperature region $T < T_c$ [10]. It means that the true solution must be somewhere beyond the replica-symmetric subspace.

3.3 Replica symmetry breaking

Since the RS solution has appeared to be not satisfactory, we should try with some other structure for the matrix $\hat{Q}$ which would contain more parameters. Within this new subspace we have to calculate the extremum of the replica free energy $f[\hat{Q}]$. After that, to check the stability of the obtained solution we have to calculate the corresponding Hessian $\delta^2 f/\delta\hat{Q}^2$.

Actually, the situation appears to be much more sophisticated since (as we will see later) no ansatz which contains finite number of parameters can provide a stable solution. Nevertheless, trying with different structures of $\hat{Q}$, and calculating the eigenvalues of the Hessian, one at least would be able to judge which ansatz could be better (so to say, which is less unstable). Such a procedure could point the correct "direction" in the space of the matrices $\hat{Q}$ towards the true solution.

The strategy of finding the true solution for the replica matrix $\hat{Q}$ in the limit $n \to 0$ is called the Parisi replica symmetry breaking (RSB) scheme [1]. This is the infinite sequence of the ansatzs which approximate the true solution better and better. Eventually, the true solution can be formulated in terms of the continuous function, which is defined as the limit of the infinite sequence. Moreover, in this limit one is able prove the stability of the obtained solution.

Consider now, step by step, which way the solution is approximated.

3.3.1 One-step RSB

At the first step, which is called the one-step RSB, it is "natural" to divide n replicas into n/m groups each containing m replicas (at this stage it is assumed that both m and n/m are integers). Then, the trial matrix $\hat{Q}$ is defined as follows: $Q_{ab} = q_1$, if the replicas a and b belong to the same group, and $Q_{ab} = q_0$, if the replicas a and b belong to different groups (the diagonal elements are equal to zeros). In the compact form such structure could be represented as follows:

$$Q_{ab} = \begin{cases} q_1 & \text{if } I(\frac{a}{m}) = I(\frac{b}{m}) \\ q_0 & \text{if } I(\frac{a}{m}) \neq I(\frac{b}{m}) \end{cases} \tag{3.23}$$

where $I(x)$ is the integer valued function, which is equal to the smallest integer bigger or equal to x. The qualitative structure of this matrix is shown in Fig.6.

In the framework of the one-step RSB we have three parameters: q_1, q_2 and m, and these parameters has to be defined from the corresponding saddle-point equations. Using the explicit form of the matrix $\hat{Q}$ for the replica free energy (3.13) one gets:

$$f[\hat{Q}] = -\frac{1}{4}\beta + \frac{1}{2n}\beta \sum_{a<b}^{n} Q_{ab}^2 - \frac{1}{\beta n} \log Z([\hat{Q}]) \tag{3.24}$$

where

$$Z([\hat{Q}]) = \sum_{\sigma_a} \exp(\beta^2 \sum_{a<b}^{n} Q_{ab}\sigma_a\sigma_b) \tag{3.25}$$

Simple algebra yields:

$$\sum_{a<b}^{n} Q_{ab}\sigma_a\sigma_b = \frac{1}{2}\left[q_0(\sum_{a=1}^{n}\sigma_a)^2 + (q_1-q_0)\sum_{k=1}^{n/m}(\sum_{c_k=1}^{m}\sigma_{c_k})^2 - nq_1\right] \tag{3.26}$$

Here k numbers the replica groups and c_k numbers the replicas inside the groups. After the Gaussian transformation in $Z[\hat{Q}]$ for each of the squares in the above equation, one gets:

$$\begin{aligned} Z[q_1,q_0,m] \;&=\; \int \tfrac{dz}{\sqrt{2\pi q_0}}\exp(-\tfrac{z^2}{2q_0})\prod_{k=0}^{n/m}\left[\int \tfrac{dy_k}{\sqrt{2\pi(q_1-q_0)}}\exp(-\tfrac{y_k^2}{2(q_1-q_0)})\right]\times \\ &\times\textstyle\sum_{\sigma_a}\exp\left(\beta z\sum_a^n\sigma_a + \beta\sum_{k=0}^{n/m}y_k(\sum_{c_k=1}^{m}\sigma_{c_k}) - \tfrac{1}{2}\beta^2 nq_1\right) \end{aligned} \tag{3.27}$$

The summation over spins yields:

$$\begin{aligned} &Z[q_1,q_0,m] \;= \\ &\exp(-\tfrac{1}{2}\beta^2 nq_1)\int\tfrac{dz}{\sqrt{2\pi q_0}}\exp(-\tfrac{z^2}{2q_0})\left[\int\tfrac{dy}{\sqrt{2\pi(q_1-q_0)}}\exp(-\tfrac{y^2}{2(q_1-q_0)})\left(2\cosh\beta(z+y)\right)^m\right]^{n/m} \end{aligned} \tag{3.28}$$

For the second term in the eq.(3.24) one obtains:

$$\frac{\beta}{2n}\sum_{a<b}^{n}Q_{ab}^2 = \frac{\beta}{4n}\left[q_1^2 m(m-1)\frac{n}{m} + q_0^2(n^2-m^2\frac{n}{m})\right] = \frac{1}{4}\beta\left[q_1^2(m-1)+q_0^2(n-m)\right] \tag{3.29}$$

Now the limit $n\to 0$ has to be taken. Originally the parameter m has been defined as an integer in the interval $1\le m\le n$. The formal analytic continuation $n\to 0$ turns this interval into $0\le m\le 1$, where m is getting to be a *continuous* parameter. Thus, taking the limit $n\to 0$ in the eqs.(3.28) and (3.29) for the free energy, eq.(3.24) one gets:

$$\begin{aligned} f(q_1,q_0,m) \;&=\; -\tfrac{1}{4}\beta\left[1+mq_0^2+(1-m)q_1^2-2q_1\right] \;- \\ -\tfrac{1}{m\beta}\textstyle\int\tfrac{dz}{\sqrt{2\pi q_0}}\exp(-\tfrac{z^2}{2q_0})\ln&\left[\int\tfrac{dy}{\sqrt{2\pi(q_1-q_0)}}\exp(-\tfrac{y^2}{2(q_1-q_0)})\left(\cosh\beta(z+y)\right)^m\right] \;-\; \ln 2 \end{aligned} \tag{3.30}$$

One can easily check that in the extreme cases $m=0$ and $m=1$ the replica symmetric solution is recovered with $q=q_0$ and $q=q_1$ correspondingly.

It should be noted that actually in the framework of the RSB formalism one has to look for the *maximum* and not for the minimum of the free energy. The formal reason is that in the limit $n\to 0$ the number of the components of the order parameter $\hat{Q}$ is getting *negative*. For example, in the case of the one-step RSB each line of the matrix $\hat{Q}$ contains $(m-1)<0$ components which are equal to q_1, and $(n-m)\to -m<0$ components which are equal to q_0. This phenomenon can also be easily demonstrated for the case when the replica free energy (3.24) would contain only the trivial term $\frac{\beta}{2n}\sum_{a<b}Q_{ab}^2$:

$$\lim_{n\to 0}\left[\frac{1}{2n}\beta\sum_{a<b}Q_{ab}^2\right] = -\frac{1}{4}\beta\left[(1-m)q_1^2+mq_0^2\right] \tag{3.31}$$

Apparently, the "correct extremum" of this free energy (in which the Hessian is positive) for $0\le m\le 1$ is the maximum and not the minimum with respect to q_0 and q_1.

To derive the saddle-point equations for the parameters q_0, q_1 and m one just has to take the corresponding derivatives of the free energy (3.30). The calculations are straightforward, but since the resulting equations are rather cumbersome, we omit this simple exercise. The results of the numerical solution of these saddle-point equations are in the following:

1) At $T < T_c = 1$ the function $f[q_1, q_0, m]$ indeed has the maximum at the non-trivial point: $0 < m(T) < 1$; $0 < q_0(T) < 1$; $0 < q_1(T) < 1$ (both for $T \to 1$ and $T \to 0$ one gets $m(T) \to 0$).

2) Although at low temperatures the entropy of this solution is getting negative again, its absolute value appears to be much smaller than that of the RS solution: $S(T = 0) \simeq -0.01$ (while for the RS solution $S(T = 0) \simeq -0.17$)

3) The most negative eigenvalue of the Hessian near T_c is equal to $-c(T - T_c)^2/9$ (c is some positive number), while for the RS solution it is equal to $-c(T - T_c)^2$. This could be interpreted, as the instability of the solution being reduced by the factor 9.

Thus, although the considered one-step RSB solution turned out to be not satisfactory too, it has appeared to be much better approximation than the RS one. Therefore one could try to move further in the chosen "direction" in the replica space.

3.3.2 Full-scale RSB

Let us try to generalize the structure of the matrix $\hat{Q}$ for more steps of the replica symmetry breaking. Let us introduce a series of integers: $\{m_i\}$ $(i = 1, 2, ..., k + 1)$ such that $m_0 = n, m_{k+1} = 1$ and all m_i/m_{i+1} at this stage are integers. Next, let us divide n replicas into n/m_1 groups such that each group would consist of m_1 replicas; each group of m_1 replicas divide into m_1/m_2 subgroups so that each subgroup would consist of m_2 replicas; and so on (Fig.7). Finally, the off-diagonal elements of the matrix $\hat{Q}$ let us define as follows:

$$Q_{ab} = q_i, \text{if } I(\tfrac{a}{m_i}) \neq I(\tfrac{b}{m_i}) \text{ and } I(\tfrac{a}{m_{i+1}}) = I(\tfrac{b}{m_{i+1}}); \quad (i = 1, 2, ..., k + 1) \tag{3.32}$$

where $\{q_i\}$ are a set of $(k + 1)$ parameters ($k = 1$ corresponds to the case of the one-step RSB).

The above definition of the matrix elements can also be represented in terms of the hierarchical tree shown in Fig.8: a particular matrix element Q_{ab} is equal to q_i corresponding to the level i of the tree, at which the lines outgoing from the points a and b meet. The structure of the matrix $\hat{Q}$ for the case $k = 2$ is shown in Fig.9.

Now we have to calculate the free energy, eqs.(3.24)-(3.25), which depends on $(k + 1)$ parameters q_i and k parameters m_i. After that, the limit $n \to 0$ has to be taken. Until the parameter n is integer, according to the above definition of the $n \times n$ matrix $\hat{Q}$ the parameters $\{m_i\}$ must satisfy the inequalities $1 \leq m_{i+1} \leq m_i \leq n$. After the analytic continuation to the limit $n \to 0$ these inequalities turn into $0 \leq m_i \leq m_{i+1} \leq 1$.

The calculation of the free energy is similar to that of the one-step RSB case. After somewhat painful algebra the result obtained for the limit $n \to 0$ is in the following:

$$\begin{aligned} f[q_0, q_1, ..., q_k; m_1, m_2, ..., m_k] \;=\; & -\tfrac{1}{4}\beta\left[1 + \textstyle\sum_{i=1}^{k}(m_{i+1} - m_i)q_i^2 - 2q_k\right] - -\tfrac{1}{m_1}\textstyle\int \frac{dz_0}{\sqrt{2\pi q_0}} \exp(-\frac{z_0^2}{2q_0})\times \\ \times \ln\{\textstyle\int dz_1 \frac{\exp(-\frac{z_1^2}{2(q_1-q_0)})}{\sqrt{2\pi(q_1-q_0)}}[\int dz_2 \frac{\exp(-\frac{z_2^2}{2(q_2-q_1)})}{\sqrt{2\pi(q_2-q_1)}}[...[\int dz_k \frac{\exp(-\frac{z_k^2}{2(q_k-q_{k-1})})}{\sqrt{2\pi(q_k-q_{k-1})}}\times \\ \times \left(2\cosh\beta(\textstyle\sum_{i=0}^{k} z_k)\right)^{m_k}]^{m_{k-1}/m_k}...]^{m_2/m_3}]^{m_1/m_2}\} \end{aligned} \tag{3.33}$$

Finally, the parameters q_i and m_i have to be obtained from the saddle-point equations:

$$\frac{\partial f}{\partial q_i} = 0\,; \quad \frac{\partial f}{\partial m_i} = 0 \tag{3.34}$$

Unfortunately, it is hardly possible to obtain the explicit analytic solutions of these equations for an arbitrary k. Nevertheless, for a given (not very large) value of k these equations can be solved numerically, and in particular, for $k = 3$ the numerical solution for the zero temperature entropy give the result $S(T = 0) \simeq -0.003$. In general, one finds that the more steps of the RSB is taken the less unstable the corresponding solution is. It indicates that presumably the true stable solution could be found in the limit $k \to \infty$. In this limit the infinite set of the parameters q_i can be described in terms

of the order parameter *function* $q(x)$ defined in the interval $(0 \leq x \leq 1)$. This function is obtained from the discrete step-like function:

$$q(x) = q_i, \quad \text{for } 0 \leq m_i < x < m_{i+1} \leq 1; \quad (i = 0, 1, ..., k) \tag{3.35}$$

in the limit of infinite number of steps, $k \to \infty$. In these terms the free energy is getting to be the functional of the function $q(x)$, and then the problem can be formulated as the searching for the maximum of this functional with respect to all (physically sensible) functions $q(x)$:

$$\frac{\delta f}{\delta q(x)} = 0 \tag{3.36}$$

For an arbitrary temperature $T < T_c$ the solution of this equation can be found only numerically. Nevertheless, near T_c all the calculations could be performed analytically, and the order parameter function $q(x)$ can be found explicitly (see Section 3.5 below). This solution appears to be quite helpful for attaining qualitative physical understanding of what is going on in the low-temperature spin-glass phase (see Chapter 4). However, before proceeding with these calculations it is necessarily to stop for a brief review of the formal general properties of the Parisi RSB matrices which will be widely used in the further considerations.

3.4 Parisi RSB algebra

Using the definitions of the previous Section one can easily prove that the linear space of the Parisi matrices, when completed with the identity $I_{ab} = \delta_{ab}$, is closed with respect to the matrix product $(QP)_{ab} = \sum_c Q_{ac} P_{cb}$ and the Hadamard product $(Q \cdot P)_{ab} = Q_{ab} P_{ab}$, operation by means of which it is possible to build polynomials which are invariant by permutations of replica indices.

Consider a generic Parisi matrix $\hat{Q}$, which in the *continuum limit* $k \to \infty$ for an *arbitrary value* of the parameter $n < 1$ is parametrized by its diagonal element $\tilde{q}$ and the off-diagonal function $q(x)$ $(n \leq x \leq 1)$: $\hat{Q} \to (\tilde{q}, q(x))$. Then for the linear invariants $\mathrm{Tr}\hat{Q}$ and $\sum_{ab} Q_{ab}$ one can easily prove:

$$\mathrm{Tr}\hat{Q} = n\tilde{q} \tag{3.37}$$

and

$$\lim_{k\to\infty} \sum_{a,b}^{n} Q_{ab} = n\tilde{q} + \lim_{k\to\infty} n \sum_{i=0}^{k} (m_i - m_{i+1}) q_i \;=\; n\tilde{q} - n \int_n^1 dx q(x) \tag{3.38}$$

Similarly to the above equation one gets:

$$\lim_{k\to\infty} \sum_{a,b}^{n} Q_{ab}^l = n\tilde{q}^l - n \int_n^1 dx q^l(x) \tag{3.39}$$

where the power l can be arbitrary.

Now let A and B be two Parisi matrices parametrized respectively by $(\tilde{a}, a(x))$ and $(\tilde{b}, b(x))$. Then for an arbitrary finite n for the Hadamard product $(Q \cdot P)_{ab} = Q_{ab} P_{ab}$ one easily proves:

$$A \cdot B \to (\tilde{a}\tilde{b}, a(x)b(x)) \tag{3.40}$$

Let us denote the parametrization of the matrix product of the two matrices as follows: $AB \to (\tilde{c}, c(x))$. Then after somewhat painful algebra one can prove that

$$\begin{aligned} &\tilde{c} = \tilde{a}\tilde{b} - \langle ab \rangle \\ &c(x) = -na(x)b(x) + (\tilde{a} - \langle a \rangle) b(x) + (\tilde{b} - \langle b \rangle) a(x) - \\ &- \textstyle\int_n^x \delta y (a(x) - a(y))(b(x) - b(y)) \end{aligned} \tag{3.41}$$

where we have introduced the notation:

$$\langle a \rangle \equiv \int_n^1 dx a(x) \tag{3.42}$$

For the eigenvalues of a Parisi matrix Q and their multiplicities one finds:

$$\begin{array}{ll} \lambda_0 = \tilde{a} - \langle a \rangle & \text{with multiplicity } 1 \\ \lambda(x) = \tilde{a} - xa(x) - \int_x^1 dy q(y) & \text{with multiplicity } -\frac{ndx}{x^2} \end{array} \tag{3.43}$$

where $x \in [n, 1]$.

The above algorithms are sufficient to operate quite easily with the Parisi matrices in the continuum RSB representation.

3.5 RSB solution near T_c

Near the critical temperature $T_c = 1$ the solution for the saddle-point function $q(x)$ can be obtained analytically. In the vicinity of the phase transition point the order parameter $q(x)$ should be expected to be small in $\tau = (T_c - T)/T_c \ll 1$, and consequently one can expand the replica free energy (3.24)-(3.25) in powers of the matrix Q_{ab}. This calculation is straightforward, and the result of the expansion up to the fourth order is in the following:

$$\begin{array}{l} f[\hat{Q}] \;=\; \lim_{n\to 0} \frac{1}{n}[-\frac{1}{2}\tau Tr(\hat{Q})^2 - \frac{1}{6}Tr(\hat{Q})^3 - \frac{1}{12}\sum_{a,b} Q_{ab}^4 + \\ +\frac{1}{4}\sum_{a,b,c} Q_{ab}^2 Q_{ac}^2 - \frac{1}{8}Tr(\hat{Q})^4] \end{array} \tag{3.44}$$

Here in all the terms but the first one we have substituted $T = 1$.

Detailed study of the stability of the replica symmetric solution shows that it is the term $\sum_{a,b} Q_{ab}^4$ which makes the RS solution to be unstable below T_c, and it is this term which is responsible for the replica symmetry breaking [9]. This indicates that for the RSB solution near T_c, the last two terms of the fourth order in (3.44) should be expected to be of higher orders in τ than all the other terms. Thus, to obtain the solution in the most easy way one can first neglect these last two terms, and then using the explicit form of the obtained solution for $q(x)$ one can easily prove *aposteriori* that these neglected terms are indeed of higher orders in τ.

Using the rules for the Parisi matrices in the continuum RSB representation described in the previous Section one can easily get the explicit expression for the free energy as the functional of $q(x)$. In particular, using eq.(3.41) for the second term in eq.(3.44) after simple algebra in the limit $n \to 0$ one gets:

$$\lim_{n\to 0} \frac{1}{n} Tr(\hat{Q})^3 \;=\; \int_0^1 dx \left[x q^3(x) + 3q(x) \int_0^x dy q^2(y) \right] \tag{3.45}$$

The first and the third terms in eq.(3.44) can be expressed using eq.(3.39) (in our case $\tilde{q} \equiv 0$). For the free energy one finally obtains:

$$f[q(x)] \;=\; \frac{1}{2} \int_0^1 dx \left[\tau q^2(x) - \frac{1}{3} x q^3(x) - q(x) \int_0^x dy q^2(y) + \frac{1}{6} q^4(x) \right] \tag{3.46}$$

Variation of this expression with respect to the function $q(x)$ yields the following saddle-point equation:

$$2\tau q(x) - x q^2(x) - 2q(x) \int_x^1 dy q(y) - \int_0^x dy q^2(y) + \frac{2}{3} q^3(x) \;=\; 0 \tag{3.47}$$

The solution of this equation is simple. Taking the derivative of eq.(3.47) over x one gets:

$$q'(x) \left[2\tau - 2xq(x) - 2 \int_x^1 dy q(y) + 2q^2(x) \right] \;=\; 0 \tag{3.48}$$

This equation results in the following:

$$2\tau - 2xq(x) - 2\int_x^1 dyq(y) + 2q^2(x) \;=\; 0 \tag{3.49}$$

or

$$q'(x) \;=\; 0 \tag{3.50}$$

The last equation means that $q(x) = const$, and it corresponds to the replica symmetric solution which has been already studied. Consider the eq.(3.49). Taking the derivative over x again, one gets:

$$q(x) \;=\; \frac{1}{2}x \tag{3.51}$$

The above simple analysis allows us to build an anzats for a general form of the solution of the original saddle-point equation (3.47):

$$q(x) \;=\; \begin{cases} q_0, & 0 \le x \le x_0 \\ \frac{1}{2}x, & x_0 \le x \le x_1 \\ q_1, & x_1 \le x \le 1 \end{cases} \tag{3.52}$$

where

$$x_1 = 2q_1 \;; \qquad x_0 = 2q_0 \tag{3.53}$$

Substituting eq.(3.52) into the original saddle-point equation (3.47) one obtains two equations for two unknown parameters q_0 and q_1:

$$\begin{aligned} q_0\left[2\tau - 2q_1 + 2q_1^2\right] - \tfrac{4}{3}q_0^3 \;&=\; 0 \\ q_1\left[2\tau - 2q_1 + 2q_1^2\right] - \tfrac{4}{3}q_0^3 \;&=\; 0 \end{aligned} \tag{3.54}$$

The solution of these equations is:

$$\begin{aligned} q_0 \;&=\; 0 \\ q_1 \;&=\; \tau + O(\tau^2) \end{aligned} \tag{3.55}$$

Now one can easily check that the last two terms of the fourth order in eq.(3.44) are of the higher order in τ compared to the other terms. It appears that since they contain additional summations over replicas, in the continuum limit representation this results in the additional integrations over x which eventually provides additional powers of τ.

Note that the obtained RSB solution could be easily generalized for the case of non-zero external magnetic field represented in the original Ising spin Hamiltonian (3.1) by the term $h\sum_i \sigma_i$. As a matter of a simple exercise one can easily derive that if the value of the field h is small in the corresponding expression for the functional RSB free energy (near T_c), eq.(3.46), the magnetic field is represented by the additional term $h^2 q(x)$. This does not change the structure of the saddle-point solution (3.52), but in the r.h.s. of the eqs.(3.54) for the parameters q_0 and q_1 one gets h^2 instead of zero. Then, in the leading order in τ and h the value of q_1 does not change, while the parameter q_0 (and x_0) is getting to be non-zero:

$$q_0 \sim h^{2/3} \tag{3.56}$$

Thus, at the critical value of the field

$$h_c(\tau) \simeq \tau^{3/2} \tag{3.57}$$

(when $x_0 = x_1$ and $q_0 = q_1$) the solution for $q(x)$ is getting to be replica symmetric. Actually, the equation for the critical line $h_c(T)$ (which is usually called the *de Almeida-Thouless* (AT) line) could be obtained for the whole range of temperatures and the magnetic fields [10]. Moreover, it can be shown that for $h > h_c(T)$ the replica symmetric solution getting stable.

4 Physics of the Replica Symmetry Breaking

In this Chapter the physical interpretation of the formal RSB solution will be proposed, and some new concepts and quantities will be introduced. The crucial concept which is needed to understand physics behind the RSB structures is that of the *pure states*.

4.1 The pure states

Consider again a simple example of the ferromagnetic system. Here below the critical temperature T_c the spontaneous symmetry breaking takes place, and at each site the non-zero spin magnetizations $\langle\sigma_i\rangle = \pm m$ appear. As we have already discussed in Section 2.2, in the thermodynamic limit the two ground states with the global magnetizations $\langle\sigma_i\rangle = +m$ and $\langle\sigma_i\rangle = -m$ are getting to be separated by an infinite energy barrier. Therefore, once the system has happened to be in one of these states, it will never be able (during any finite time) to jump into the other one. In this sense, the *observable* state is not the Gibbs one (which is obtained by summing over all the states), but one of these two states with non-zero global magnetizations. To distinguish them from the Gibbs state they could be called the "pure states". More formally the pure states could also be defined by the property that all the connected correlation functions in these states, such as $\langle\sigma_i\sigma_j\rangle_c \equiv \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$, are getting to be zero at large distances.

In the previous Chapter we have obtained a special type of the spin-glass ground state solution. Formally this solution is characterized by the RSB in the corresponding order parameter matrix Q_{ab}. It means that actually there exist many other solutions of this type in the spin-glass phase. This fact is a direct consequence of the symmetry of the replica free energy (3.24)-(3.25) with respect to permutations of replicas: if there exists a particular solution for the matrix $\hat{Q}_*$ with the RSB, then any other matrix obtained via permutations of the replica indices in $\hat{Q}_*$ will also be a solution. On the other hand, since the total mean-field free energy (which is the function of $\hat{Q}$) is proportional to the volume of the system the energy barriers separating the corresponding ground states must be getting infinite in the thermodynamic limit. Consequently, just like in the example of the ferromagnetic system, all these RSB states could called the pure states of the low-temperature spin-glass phase. Correspondingly, the Gibbs state of the spin glass (which is formally obtained by summing over *all* the states of the system) could be considered as being given by the summation over all the pure states with the corresponding thermodynamic weight defined by values of their free energies.

For instance, the thermodynamic (Gibbs) average of the site magnetizations could be represented as follows:

$$\langle\sigma_i\rangle \equiv m_i = \sum_\alpha w_\alpha m_i^\alpha \tag{4.1}$$

Here m_i^α are the site magnetizations in the pure state number α, and w_α denotes its statistical weight which formally could be represented as follows:

$$w_\alpha = \exp(-F_\alpha) \tag{4.2}$$

where F_α is the free energy corresponding to this pure state. In the same way the two-point correlation function can be represented as the linear combination

$$\langle\sigma_1\sigma_2\rangle = \sum_\alpha w_\alpha \langle\sigma_1\sigma_2\rangle_\alpha \tag{4.3}$$

where $\langle\sigma_1\sigma_2\rangle_\alpha$ is the two-point correlation function in the pure state number α. According to the definition of the pure state

$$\langle\sigma_1\sigma_2\rangle_\alpha = \langle\sigma_1\rangle_\alpha \langle\sigma_2\rangle_\alpha \tag{4.4}$$

Similar expressions could be written for any many-point correlation functions.

The representation of the thermodynamic Gibbs state as a linear combination of the pure states in which all extensive quantities have vanishing long-distance fluctuations, is actually, a central point in the exact definition of the concept of the spontaneous symmetry breaking in statistical mechanics.

4.2 Physical order parameter $P(q)$ and the replica solution

To investigate the statistical properties of the spin-glass pure states let us define the *overlaps* $\{q_{\alpha\beta}\}$ among them as follows:

$$q_{\alpha\beta} \equiv \frac{1}{N}\sum_{i}^{N} m_i^{\alpha} m_i^{\beta} \tag{4.5}$$

where $m_i^{\alpha} = \langle\sigma_i\rangle_{\alpha}$ and $m_i^{\beta} = \langle\sigma_i\rangle_{\beta}$ are the site magnetizations in the pure states α and β. Apparently, $0 \leq \mid q_{\alpha\beta} \mid \leq 1$.

To describe the statistics of these overlaps it is natural to introduce the following the probability distribution function:

$$P_J(q) = \sum_{\alpha\beta} w_{\alpha} w_{\beta} \delta(q_{\alpha\beta} - q) \tag{4.6}$$

Note, that this distribution function is defined for a given sample, and it can depend on a concrete realization of the quenched interactions J_{ij}. The "observable" distribution function should, of course, be averaged over the disorder parameters:

$$P(q) = \overline{P_J(q)} \tag{4.7}$$

The distribution function $P(q)$ gives the probability to find two pure states having the overlap equal to q, conditioned that these states are taken with their statistical thermodynamic weights $\{w_{\alpha}\}$.

It is the distribution function $P(q)$ which can be considered as the physical order parameter. It should be stressed that $P(q)$ is much more general concept than ordinary order parameters which usually describe the phase transitions in ordinary statistical systems. The fact that it is a *function* is actually a manifestation of the crucial phenomenon that for the description of the spin glass phase one needs an infinite number of the order parameters. The non-trivial structure of this distribution function (it will be calculated explicitly below) demonstrates that the properties of the spin glass state are essentially different from those of the traditional magnets.

Consider now which way the order parameter function $P(q)$ could be calculated in terms of the replica method. Let us introduce the following set of the correlation functions:

$$\begin{aligned} q_J^{(1)} &= \tfrac{1}{N}\textstyle\sum_i \langle\sigma_i\rangle^2 \\ q_J^{(2)} &= \tfrac{1}{N^2}\textstyle\sum_{i_1 i_2} \langle\sigma_{i_1}\sigma_{i_2}\rangle^2 \\ &\cdots\cdots\cdots\cdots \\ q_J^{(k)} &= \tfrac{1}{N^k}\textstyle\sum_{i_1 \ldots i_k} \langle\sigma_{i_1}\ldots\sigma_{i_k}\rangle^2 \end{aligned} \tag{4.8}$$

Using the representation of the Gibbs averages in terms of the pure states (4.3)-(4.4) for the correlation functions (4.8) one gets:

$$q_J^{(1)} \;=\; \tfrac{1}{N}\sum_i(\sum_\alpha w_\alpha\langle\sigma_i\rangle_\alpha)(\sum_\beta w_\beta\langle\sigma_i\rangle_\beta) \;=$$

$$=\; \sum_{\alpha\beta} w_\alpha w_\beta q_{\alpha\beta} \;=\; \int dqP_J(q)q \quad ;$$

$$q_J^{(2)} \;=\; \tfrac{1}{N^2}\sum_{i_1 i_2}(\sum_\alpha w_\alpha\langle\sigma_{i_1}\sigma_{i_2}\rangle_\alpha)(\sum_\beta w_\beta\langle\sigma_{i_1}\sigma_{i_2}\rangle_\beta) \;=$$

$$=\; \sum_{\alpha\beta} w_\alpha w_\beta(\tfrac{1}{N}\sum_{i_1}\langle\sigma_{i_1}\rangle_\alpha\langle\sigma_{i_1}\rangle_\beta)(\tfrac{1}{N}\sum_{i_2}\langle\sigma_{i_2}\rangle_\alpha\langle\sigma_{i_2}\rangle_\beta) \;= \tag{4.9}$$

$$=\; \sum_{\alpha\beta} w_\alpha w_\beta(q_{\alpha\beta})^2 \;=\; \int dqP_J(q)q^2 \quad ;$$

.....

$$q_J^{(k)} \;=\; \int dqP_J(q)q^k$$

For the corresponding correlation functions averaged over the disorder from eqs.(4.8)-(4.9) one gets:

$$q^{(1)} \equiv \overline{q_J^{(1)}} \;=\; \overline{\langle\sigma_i\rangle^2} \;=\; \int dqP(q)q$$

..... (4.10)

$$q^{(k)} \;\equiv\; \overline{q_J^{(k)}} \;=\; \overline{\langle\sigma_{i_1}...\sigma_{i_k}\rangle^2} \;=\; \int dqP(q)q^k$$

where $i_1 \neq i_2 \neq ... \neq i_k$.

The crucial point in the above consideration is that the function $P(q)$ originally defined to describe the statistics of (somewhat abstract) pure states, can be calculated (at least theoretically) from the multipoint correlation functions in the *Gibbs states*. Therefore, if we could be able to calculate the above multipoint correlation functions in terms of the replica approach, the connection of the formal RSB scheme with the physical order parameter would be established.

In terms of the replica approach the correlator $q^{(1)} = \overline{\langle\sigma_i\rangle^2}$ can be represented as follows:

$$q^{(1)} \;=$$

$$\overline{\tfrac{1}{Z^2}\sum_\sigma\sum_s(\sigma_i s_i)\exp(-\beta H[\sigma]-\beta H[s])} \;= \tag{4.11}$$

$$=\; \lim_{n\to 0}(\textstyle\prod_{a=1}^n\sum_{\sigma^a})(\sigma_i^b\sigma_i^c)\exp(-\beta\sum_{a=1}^n H[\sigma^a])$$

$$\equiv\; \lim_{n\to 0}\overline{\langle\sigma_i^b\sigma_i^c\rangle} \qquad (b\neq c)$$

where a and b are two different replicas (the summation over the rest $(n-2)$ replicas in eq.(4.11) gives the factor Z^{n-2} which turns into Z^{-2} in the limit $n\to 0$). In a similar way one gets:

$$q^{(2)} \;=\; \lim_{n\to 0}\overline{\langle\sigma_{i_1}^a\sigma_{i_2}^a\sigma_{i_1}^b\sigma_{i_2}^b\rangle} \;;\;\; (i_1\neq i_2;\; a\neq b)$$

..... (4.12)

$$q^{(k)} \;=\; \lim_{n\to 0}\overline{\langle\sigma_{i_1}^a...\sigma_{i_k}^a\sigma_{i_1}^b...\sigma_{i_k}^b\rangle} \;;\;\; (i_1\neq i_2\neq ...\neq i_k;\; a\neq b)$$

In the calculations of the previous Chapter it has been demonstrated that the free energy of the model under consideration is factorizing into the independent site replica free energies. Therefore, the result (4.12) for $q^{(k)}$ can be represented as follows:

$$q^{(k)} \;=\; \lim_{n\to 0}[\overline{\langle\sigma_i^a\sigma_i^b\rangle}]^k \;=\; \lim_{n\to 0}[Q_{ab}]^k \tag{4.13}$$

where (see eq.(3.9))

$$Q_{ab} \;=\; \overline{\langle \sigma_i^a \sigma_i^b \rangle} \tag{4.14}$$

is the replica order parameter matrix introduced in the Chapter 3, which is obtained from the saddle point equation for the replica free energy. Since in the RSB solution the matrix elements of Q_{ab} are not equivalent, for performing the Gibbs average one has to sum over *all* the saddle point solutions for the matrix Q_{ab}. Such solutions can be obtained from one of the RSB solutions by doing all possible permutations of rows and columns in Q_{ab}. The summation over all these permutations corresponds to the summation over the replica subscripts a and b of the matrix Q_{ab}. Thus, the final result for the correlator $q^{(k)}$ should be represented as follows:

$$q^{(k)} \;=\; \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} [Q_{ab}]^k \tag{4.15}$$

where $n(n-1)$ is the normalization factor which is equal to the number of different replica permutations.

The results (4.15) and (4.10) demonstrate that using the formal RSB solution for the matrix Q_{ab} considered in the previous Chapter one can calculate the order parameter distribution function $P(q)$ which has been originally introduced on the basis of purely qualitative physical arguments. From these two equations one gets the following explicit expression for the distribution function $P(q)$:

$$P(q) \;=\; \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} \delta(Q_{ab} - q) \tag{4.16}$$

Using the algorithms of the Parisi algebra, Section 3.4, in the continuous RSB representation this result can be rewritten as follows:

$$P(q) = \int_0^1 dx \delta(q(x) - q) \tag{4.17}$$

Assuming that the function $q(x)$ is monotonous (which is the case for the saddle-point solution obtained in Chapter 3), one can introduce the inverse function $x(q)$, and then from eq.(4.17) one finally obtains:

$$P(q) \;=\; \frac{dx(q)}{dq} \tag{4.18}$$

(Note that the same result can be obtained by comparing the eqs.(4.15) and (4.10).) This is key result, which defines the physical order parameter distribution function $P(q)$ in terms of the formal saddle-point Parisi function $q(x)$.

The above result can also be represented in the integral form:

$$x(q) \;=\; \int_0^q dq' P(q') \tag{4.19}$$

which gives the answer to the question, what is the physical meaning of the Parisi function $q(x)$. According to eq.(4.19) the answer is in the following: the function $x(q)$ inverse to $q(x)$ gives the probability to find a pair of the pure states which would have the overlap not bigger than q

Using the explicit solution for the Parisi function $q(x)$ in the vicinity of the critical point, eqs.(3.52)-(3.56), according to eq.(4.18) for the distribution function $P(q)$, one gets:

$$P(q) \;=\; x_0 \delta(q - q_0) + (1 - x_1)\delta(q - q_1) + p(q) \tag{4.20}$$

where $p(q)$ is the smooth function defined in the interval $q_0 \leq q \leq q_1$. In the close vicinity of the critical point, $\tau \ll 1$, where the solution (3.52) is valid, this function is just constant: $p(q) = 2$.

The result (4.20) shows that the statistics of the overlaps of the pure states demonstrates the following properties:

1) There is a finite probability $(1 - x_1) \simeq (1 - 2\tau)$ that taken at random two pure states would appear to be the same state. The "selfoverlaps", eq.(2.3), of these states is equal to $q_1 \simeq \tau$.

2) In the presence of non-zero external magnetic field h there is a finite probability $x_0 \sim h^{2/3}$ that taken at random two pure states would appear to be the most "distant" having the minimum possible overlap $q_0 \sim h^{2/3}$.

3) there is a finite probability $(x_1 - x_0)$ that taken at random two pure states would have the overlap q in the interval $q_0 \leq q \leq q_1$. For a given small interval δq there is a finite probability $p(q)\delta q$ to find two pure states with the overlap in the interval $(q, q + \delta q)$, where $q_0 \leq q \leq q_1$.

Although for arbitrary values of the temperature and the magnetic field it is hardly possible to calculate the functions $q(x)$ and $P(q)$ analytically, their qualitative behavior remains similar to the case considered above. The only difference is that the concrete shape of the function $P(q)$ in the interval $q_0 \leq q \leq q_1$ (as well as the function $q(x)$ at the interval $x_0 \leq x \leq x_1$) is getting less trivial. Besides the dependencies of x_0, x_1, q_0 and q_1 from the temperature and the magnetic field are getting more complicated.

The qualitative behavior of the functions $q(x)$ and $P(q)$ for different values of the temperature and the magnetic field are shown in Fig.10.

5 Ultrametricity

5.1 Ultrametric structure of pure states

The solutions for the functions $q(x)$ and $P(q)$, obtained in the previous Chapters, indicate that the structure of the space of the spin-glass pure states must be highly non-trivial. However, the distribution function $P(q)$ of the pure states overlaps does not give enough information about this structure. To get insight into the topology of the space of the pure states one needs to know the properties of the higher order correlations of the overlaps.

Let us consider the distribution function $P(q_1, q_2, q_3)$ which describes the joint statistics of the overlaps of arbitrary *three* pure states. By definition, for arbitrary three pure states α, β and γ this function gives the probability that their mutual overlaps $q_{\alpha\beta}, q_{\alpha\gamma}$ and $q_{\beta\gamma}$ are equal correspondingly to q_1, q_2 and q_3:

$$P(q_1, q_2, q_3) \;=\; \overline{\sum_{\alpha\beta\gamma} w_\alpha w_\beta w_\gamma \delta(q_1 - q_{\alpha\beta})\delta(q_2 - q_{\alpha\gamma})\delta(q_3 - q_{\beta\gamma})} \tag{5.1}$$

In terms of the RSB scheme the calculation of this function is quite similar to that for the function $P(q)$. In particular, in terms of the replica matrix Q_{ab} instead of the eq.(4.16), in the present case one can easily prove that

$$\begin{array}{l} P(q_1, q_2, q_3) \;= \\ \lim_{n\to 0} \frac{1}{n(n-1)(n-2)} \sum_{a\neq b\neq c} \delta(Q_{ab} - q_1)\delta(Q_{ac} - q_2)\delta(Q_{bc} - q_3) \end{array} \tag{5.2}$$

In terms of the Fourier transform of the function $P(q_1, q_2, q_3)$:

$$g(y_1, y_2, y_3) \;=\; \int dq_1 dq_2 dq_3 P(q_1, q_2, q_3) \exp(iq_1 y_1 + iq_2 y_2 + iq_3 y_3) \tag{5.3}$$

instead of eq.(5.2) one gets:

$$\begin{array}{l} g(y_1, y_2, y_3) \;= \\ \lim_{n\to 0} \frac{1}{n(n-1)(n-2)} \sum_{a\neq b\neq c} \exp(iQ_{ab} y_1 + iQ_{ac} y_2 + iQ_{bc} y_3) \;= \\ \lim_{n\to 0} \frac{1}{n(n-1)(n-2)} Tr[\hat{A}(y_1)\hat{A}(y_2)\hat{A}(y_3)] \end{array} \tag{5.4}$$

where

$$A_{ab}(y) \;= \left\{ \begin{array}{ll} \exp(iQ_{ab}y) \;; & a \neq b \\ 0 \;; & a = b \end{array} \right. \tag{5.5}$$

Let us substitute the RSB solution for the matrix Q_{ab} into the eq.(5.4). In the continuum RSB limit the matrix Q_{ab} turns into the function $q(x)$, and according to the Parisi algebra (Section 3.4) the replica matrix $A_{ab}(y)$ turns into the corresponding function $A(x; y)$:

$$A(x;y) \;=\; \exp(iq(x)y) \tag{5.6}$$

Using the algorithms of the Parisi algebra, eqs.(3.39)-(3.43) after simple calculations one obtains:

$$\begin{aligned}
&\lim_{n\to 0} \tfrac{1}{n(n-1)(n-2)} Tr[\hat{A}(y_1)\hat{A}(y_2)\hat{A}(y_3)] \;= \\
&= \tfrac{1}{2}\int_0^1 dx[xA(x;y_1)A(x;y_2)A(x;y_3) + A(x;y_1)\int_0^x dzA(z;y_2)A(z;y_3) + \\
&A(x;y_2)\int_0^x dzA(z;y_1)A(z;y_3) + A(x;y_3)\int_0^x dzA(z;y_1)A(z;y_2)]
\end{aligned} \tag{5.7}$$

Accordingly, for the function $P(q_1,q_2,q_3)$:

$$P(q_1,q_2,q_3) \;=\; \int dy_1 dy_2 dy_3 g(y_1,y_2,y_3)\exp(-iq_1y_1 - iq_2y_2 - iq_3y_3) \tag{5.8}$$

one gets:

$$\begin{aligned}
&P(q_1,q_2,q_3) \;= \\
&= \tfrac{1}{2}\int_0^1 dx[x\delta(q(x)-q_1)\delta(q(x)-q_2)\delta(q(x)-q_3) + \\
&\delta(q(x)-q_1)\int_0^x dz\delta(q(z)-q_2)\delta(q(z)-q_3) + \\
&\delta(q(x)-q_2)\int_0^x dz\delta(q(z)-q_1)\delta(q(z)-q_3) + \\
&\delta(q(x)-q_3)\int_0^x dz\delta(q(z)-q_1)\delta(q(z)-q_2)
\end{aligned} \tag{5.9}$$

Introducing the integration over q instead of that over x and taking into account that $dx(q)/dq = P(q)$ one finally obtains the following result:

$$\begin{aligned}
&P(q_1,q_2,q_3) \;= \tfrac{1}{2}P(q_1)x(q_1)\delta(q_1-q_2)\delta(q_1-q_3) + \\
&\tfrac{1}{2}P(q_1)P(q_2)\theta(q_1-q_2)\delta(q_2-q_3) + \\
&\tfrac{1}{2}P(q_2)P(q_3)\theta(q_2-q_3)\delta(q_3-q_1) + \\
&\tfrac{1}{2}P(q_3)P(q_1)\theta(q_3-q_1)\delta(q_1-q_2)
\end{aligned} \tag{5.10}$$

From this equation one can easily see the following crucial property of the function $P(q_1,q_2,q_3)$. It is non-zero only in the following three cases: $q_1 = q_2 \le q_3$; $\quad q_1 = q_3 \le q_2$; $\quad q_3 = q_2 \le q_1$. In all other cases the function $P(q_1,q_2,q_3)$ is identically equal to zero. In other words, this function is not equal to zero only if at least two of the three overlaps are equal, and their value is not bigger than the third one. It means that in the space of spin glass states there exist no triangles with all three sides being different. The spaces having the above metric property are called *ultrametric*.

A simple illustration the ultrametric space can be given in terms of the hierarchical tree (Fig.11). The ultrametric space here is associated with the set of the endpoints of the tree. By definition, the overlaps between any two points of this space depends only on the number of "generations" (in the "vertical" direction) to the level of the tree where these two points have a common ancestor. One can easily check that paired overlaps among arbitrary three points of this set do satisfy the above ultrametric property.

A detailed description of the ultrametric spaces the reader can find in the review [7]. Here we are going to concentrate only on a general qualitative properties of the ultrametricity which are crucial for the physics of the spin glass state.

5.2 The tree of states

Let us consider how the spin-glass ultrametric structures can be defined in more general terms.

Consider the following discrete stochastic process which is assumed to take place *independently* at each site i of the lattice.

1. At the first step, with the probability $P_0(y)$ one generates n_1 random numbers y^{α_1} $(\alpha_1 = 1, 2, ..., n_1)$, which belong to the interval $[-1, +1]$.

2. At the second step, for each y^{α_1} with the conditional probability $P_1(y^{\alpha_1}|y)$ one generates n_2 random numbers $y^{\alpha_1\alpha_2}$ $(\alpha_2 = 1, 2, ..., n_2)$, belonging to the same interval $[-1, +1]$.

3. At the third step, for each $y^{\alpha_1\alpha_2}$ with the conditional probability $P_2(y^{\alpha_1\alpha_2}|y)$ one generates n_3 random numbers $y^{\alpha_1\alpha_2\alpha_3}$ $(\alpha_3 = 1, 2, ..., n_3)$, belonging to the same interval $[-1, +1]$.

.....

This process is continued up to the L-th step. Finally, in the interval $[-1, +1]$ one gets $n_1 n_2 ... n_L$ random numbers, which are described by the following set of the probability functions

$$P_{l-1}(y^{\alpha_1...\alpha_{l-1}}|y^{\alpha_1...\alpha_l}) \qquad (l = 1, 2, ..., L) \tag{5.11}$$

This stochastic (Markov) process takes place independently at each site of the lattice. Then, for each set of the obtained random numbers let us define the corresponding site spin states as follows:

$$\sigma_i^{\alpha_1...\alpha_L} \;=\; sign(y_i^{\alpha_1...\alpha_L}) \tag{5.12}$$

This way one obtains the set of $n_1 n_2 ... n_L$ spin states which are labeled by the hierarchical "address" $\alpha_1 ... \alpha_L$. The "address" of a concrete state describes its genealogical "history".

Simple probabilistic arguments show that the overlap between any two spin states depends only on the degree of their "relativeness", i.e. it is defined only by the number of generations which separates them from the closest common ancestor. Consider two spin states which have the following "addresses":

$$\alpha_1\alpha_2...\alpha_l\alpha_{l+1}\alpha_{l+2}...\alpha_L$$

and

$$\alpha_1\alpha_2...\alpha_l\beta_{l+1}\beta_{l+2}...\beta_L$$

These two "addresses" are getting different starting from the generation number l. Since the stochastic processes generating the states is independent at each site, for the overlap between these two states

$$q_{\alpha_1...\alpha_l\beta_{l+1}...\beta_L}^{\alpha_1...\alpha_l\alpha_{l+1}...\alpha_L} \;=\; \frac{1}{N}\sum_i^N \sigma_i^{\alpha_1...\alpha_l\alpha_{l+1}...\alpha_L}\sigma_i^{\alpha_1...\alpha_l\beta_{l+1}...\beta_L} \tag{5.13}$$

in the thermodynamic limit $N \to \infty$ one gets:

$$\begin{aligned} & q_{\alpha_1...\alpha_l\beta_{l+1}...\beta_L}^{\alpha_1...\alpha_l\alpha_{l+1}...\alpha_L} \;= \\ & \textstyle\int_{-1}^{+1} dy_1...dy_l P_0(y_1)P_1(y_1|y_2)...P_{l-1}(y_{l-1}|y_l)\times \\ & \times[\textstyle\int_{-1}^{+1} dy_{l+1}...dy_L P_l(y_l|y_{l+1})P_{l+1}(y_{l+1}|y_{l+2})...P_{L-1}(y_{L-1}|y_L)sign(y_L)]^2 \;\equiv\; q_l \end{aligned} \tag{5.14}$$

Therefore, the overlap depends only on the number l of the level of the tree at which the two states were separated in their genealogical history, and does not depend on the concrete "addresses" of these states. One can easily see that it automatically means that the considered set of the states is ultrametric.

Note, that this is a general property of the considered stochastic evolution process, and it remains to be true for any choice of the probability distribution functions (5.11) which describe the concrete tree of states. A general reason for that is very simple. The above stochastic procedure has been defined as the random branching process which takes place in the infinite dimensional space (in the limit $N \to \infty$), and it is clear that here the branches once separated never comes close again. Therefore, it is of no surprise that the ultrametricity is observed in Nature very often. The examples are the space of biological species, the hierarchical state structures of disordered human societies, etc.

Let us consider the above hierarchical tree of states in some more details. The equations for the overlaps between two spin states (5.13) and (5.14) can also be represented in terms of the so-called *ancestor states* $m^{\alpha_1...\alpha_l}$:

$$q_l \;=\; \frac{1}{N}\sum_i^N (m_i^{\alpha_1...\alpha_l})^2 \tag{5.15}$$

where the site magnetizations in the ancestor state $m^{\alpha_1...\alpha_l}$ at the level l are defined as follows:

$$m_i^{\alpha_1...\alpha_l} \;=\; \langle \sigma_i^{\alpha_1...\alpha_l\alpha_{l+1}...\alpha_L}\rangle_{(\alpha_{l+1}...\alpha_L)} \;\equiv\; m_l(y_i^{\alpha_1...\alpha_l}) \tag{5.16}$$

Here $\langle ...\rangle_{(\alpha_{l+1}...\alpha_L)}$ denotes the averaging over all the descendant states (branches) of the tree outgoing from the branch $\alpha_1...\alpha_l$ at the level number l. By definition:

$$\begin{aligned} & m_l(y_i^{\alpha_1...\alpha_l}) \;= \\ & = \int_{-1}^{+1} dy_{l+1}...dy_L P_l(y_i^{\alpha_1...\alpha_l}|y_{l+1})P_{l+1}(y_{l+1}|y_{l+2})...P_{L-1}(y_{L-1}|y_L)sign(y_L) \end{aligned} \tag{5.17}$$

This equation for the function $m_l(y)$ could also be written in the following recurrent form:

$$m_l(y) \;= \int_{-1}^{+1} dy' \mathbf{P}_{ll'}(y|y')m_{l'}(y') \tag{5.18}$$

where

$$\mathbf{P}_{ll'}(y|y') \;=\; \int_{-1}^{+1} dy_{l+1}...dy_{l'-1}P_l(y|y_{l+1})P_{l+1}(y_{l+1}|y_{l+2})...P_{l'-1}(y_{l'-1}|y') \tag{5.19}$$

Therefore, all the concrete properties of the tree of states, and in particular the values of the overlaps $\{q_l\}$, are fully determined by the set of the probability functions (5.11) or (5.19). For the complete description of a concrete spin glass system all these functions have to be calculated, or at least the algorithms of their calculations must be derived. In particular, this can be done for the SK model of spin glass. Unfortunately, the corresponding calculations for this model are rather cumbersome, and the reader interested in the details may refer to the original papers [13] and [14]. Here only the final results will be presented.

The ultrametric tree of states which describes the spin glass phase of the SK model is defined by the random branching process described above, in which the continuous limit $L \to \infty$ must be taken. In this limit, instead of the integer numbers l which define the discrete levels of the hierarchy, it is more convenient to describe the tree in terms of the selfoverlaps $\{q_l\}$ of the ancestor states. In the limit $L \to \infty$ the discrete parameters $\{q_l\}$ are getting to be the continuous variable $0 \le q \le 1$.

Instead of the discrete "one-step" functions (5.11) in the continuous limit it is more natural to describe the tree in terms of the functions (5.19) which define the evolution of the tree from the level q to the other level q'. It can be proved (an it is this proof which requires to go through somewhat painful algebra) that in the continuous limit these functions are defined by the following non-linear diffusion equation:

$$-\frac{\partial}{\partial q}\mathbf{P} \;=\; \frac{1}{2}\frac{\partial^2}{\partial y^2}\mathbf{P} \;+\; x(q)m_q(y)\frac{\partial}{\partial y}\mathbf{P} \tag{5.20}$$

with the initial condition:

$$\lim_{q\to q'} \mathbf{P}_{qq'}(y|y') \;=\; \delta(y-y') \tag{5.21}$$

Here $x(q)$ is the function inverse to $q(x)$ (which is given by the RSB solution, Chapter 3), and the function $m_q(y)$ is the continuous limit of the discrete function (5.18). It can be shown that this function defines the distribution of the site magnetizations in the ancestor states at the level q of the tree. One can easily derive from the eqs. (5.18) and (5.20) that the function $m_q(y)$ satisfies the following equation:

$$-\frac{\partial}{\partial q}m_q(y) \;=\; \frac{1}{2}\frac{\partial^2}{\partial y^2}m_q(y) \;+\; x(q)m_q(y)\frac{\partial}{\partial y}m_q(y) \tag{5.22}$$

The above equations fully describe the properties of the ultrametric tree of the spin-glass states of the SK model.

5.3 Scaling in a space of spin-glass states

Let us summarize all the results obtained for the spin glass model with the long range interactions:

1) In terms of the formal replica calculations the free energy of the system can be represented it in terms of the functional $F[\hat{Q}]$ which depends on the $n \times n$ replica matrix $\hat{Q}$ (Section 3.1). In the thermodynamic limit the leading contribution to the free energy comes from the matrices $\hat{Q}^*$ which correspond to the extrema of this functional, and the physical free energy is obtained in the limit $n \to 0$. In this limit the extrema matrices $\hat{Q}^*$ are defined by the infinite set of parameters which can be described in terms of the continuous Parisi function $q(x)$ defined at the interval $0 \leq x \leq 1$ (Sections 3.3 - 3.4). In the low-temperature region near the phase transition point this function can be obtained explicitly (Section 3.5, Fig.10).

2) On the other hand, in terms of qualitative physical arguments one can define as the order parameter the distribution function $P(q)$ which gives the probability to find a pair of pure spin glass states having the overlap equal to q. In terms of the RSB scheme one can show that the distribution function $P(q)$ is defined by the Parisi function $q(x)$: $P(q) = dx(q)/dq$, where $x(q)$ is the inverse function to $q(x)$ (Section 4.2). The low-temperature solutions for $q(x)$ and for $P(q)$ show that there exists the continuous spectrum of the overlaps among the pure states.

3) Next, one can introduce the "three-point" distribution function $P(q_1, q_2, q_3)$ which gives the probability that arbitrary three pure states have their mutual pair overlaps equal to q_1, q_2 and q_3. In terms of the RSB scheme this function can be calculated explicitly, and the obtained result show that the space of the pure states has the ultrametric topology (Section 5.1).

4) It can be shown that the ultrametric tree-like structures can be described in terms of the hierarchical evolution tree which is defined by the random branching process.

Basing on the above results, the spin-glass phase can be described in the qualitative physical terms as follows (see also Chapter 2).

At a given temperature T below T_c the space of spin states is splitted into numerous pure states (valleys) separated by infinite energy barriers. Although the average site magnetizations m_i are different in different states, the value of the selfoverlap:

$$q(T) \; = \; \sum_{i}^{N} m_i^2 \tag{5.23}$$

appears to be the same in all the states. The value of q is the function of the temperature ($q(T_c) = 0$; $q(0) = 1$), and near T_c it can be calculated explicitly.

On the other hand, the overlaps $q^{\alpha\beta}$ of the pure states cover continuously the whole interval $0 \leq q^{\alpha\beta} \leq q(T)$. (In the presence of external magnetic field h this interval starts from non-zero value: $q_0(h,T) \leq q^{\alpha\beta} \leq q_1(h,T)$). The distribution of the values of the overlaps $q^{\alpha\beta}$ is described by a probability function $P(q)$ which depends on the temperature (and the magnetic field). The structure of the space of the pure states can be described in terms of the ultrametric hierarchical tree discussed above.

Now, if the temperature is slightly decreased $T \to T' = T - \delta T$, each of the pure states is splitted into numerous new "descendant" pure states. These states are characterized by the new value of the selfoverlap $q(T') > q(T)$. Correspondingly, the interval of their overlaps is getting bigger: $0 \leq q^{\alpha\beta} \leq q(T')$.

At further decrease of the temperature each of the newly borne pure states is splitted again into new descendant pure states, and this branching process continues down to zero temperature ($q(T \to 0) \to 1$). The tree of pure states obtained this way has the property of the self-similarity (scaling), and at any given temperature the natural scale in the space of states is given by the value of $q(T)$.

Due to infinite energy barriers separating the valleys the "observable" physics at the given temperature T is defined by only one of the pure states, which in terms of the hierarchical tree corresponds to one of the "ancestor" states at the level (scale) $q(T)$. All these states could be obtained in the horizontal crossection of the tree at the level $q(T)$.

5.4 Phenomenological dynamics

Although the dynamical properties of spin-glasses is extremely hard problem even at the mean-field level, certain (the most simple) general slow relaxation properties of the disordered systems with the hierarchical structure of the free energy landscape can be understood rather easily using purely phenomenological approach [15].

Assume that the free energy landscape in the spin glass phase is of the type shown in Fig.3: big wells contain a lot of smaller ones, each of the smaller wells contains a lot of even smaller ones, and so on. Such kind of the landscape could be characterized by the typical value of the *finite* energy barrier $\Delta(q)$ separating the wells at the scale q. Assuming that this landscape has the scaling property, the dependence of the typical value of the energy barrier Δ from the scale q could be described by the following simple scaling law:

$$\Delta(q) \;=\; \Delta_0(q - q(T))^{-\nu} \;\; ; \qquad (q > q(T) \;\; ; \;\; \nu > 0) \tag{5.24}$$

Here $q(T)$ is the value of the selfoverlap of the pure states at the temperature T. The parameter $q(T)$ is the characteristic scale (the typical scale of the valleys) at which the barriers separating the states are getting infinite.

Consider now what kind of the relaxation properties could be derived from the above assumptions. The characteristic time needed to overcome the barrier Δ is

$$\tau(\Delta) \sim \tau_0 \exp(\frac{\Delta}{T}) \tag{5.25}$$

where τ_0 is characteristic microscopic time. Thus, the spectrum of the relaxation times inside the valley can be represented as follows:

$$\tau(q) \sim \tau_0 \exp[\beta\Delta_0(q - q(T))^{-\nu}] \tag{5.26}$$

Then the long-time relaxation behaviour of the order parameter

$$q(t) \;=\; \frac{1}{N}\sum_i \langle\sigma_i(0)\sigma_i(t)\rangle \tag{5.27}$$

can be estimated (very roughly) as follows:

$$q(t) \;\sim\; \int_{q(T)}^{1} dq \; q \exp(-\frac{t}{\tau(q)}) \tag{5.28}$$

Using (5.26), one gets:

$$q(t) \;\sim\; \int_{q(T)}^{1} dq \; \exp\left(\ln(q) - \frac{t}{\tau_0}\exp[-\beta\Delta_0(q - q(T))^{-\nu}]\right) \tag{5.29}$$

In the limit of large times $t >> \tau_0$ the saddle-point estimate of the above integral gives the following result:

$$q(t) \sim \; q(T) \; + \; [\frac{\beta\Delta_0}{\ln(t/\tau_0)}]^{\frac{1}{\nu}} \tag{5.30}$$

Therefore at large times the order parameter approaches its equilibrium value $q(T)$ logarithmically slowly. Apparently, the relaxation behavior of others observable quantities should be of the same slow type.

Of course, true dynamic properties of spin-glasses are much more complicated, and they can not be reduced only to the phenomenon of extremely slow relaxation. Actually, the main property of spin-glasses is that they can not reach true thermodynamic equilibrium at any finite observation time. Since the theoretical achievements in understanding of the dynamical properties of spin-glasses are far from being quite impressive yet, in the next Chapter we consider the results of the experimental observations of the relaxation phenomena in real spin-glass magnets.

6 Experiments

In this Chapter we will consider classical experiments which have been performed on *real* spin glass materials, aiming to check to what extent the qualitative picture of the spin-glass state described in previous Chapters does take place in real world. The main problem of the experimental observations is that the concepts and quantities which are very convenient in theoretical considerations are rather far from the experimental realities, and it is a matter of the experimental art to invent convincing experimental procedures which would be able to confirm (or reject) the theoretical predictions.

A series of such brilliant experiments has been performed by M.Ocio, J.Hammann, F.Lefloch and E.Vincent (Saclay), and M.Lederman and R.Orbach (UCLA) [16]. Most of these experiments have been done on the crystals $CdCr_{1.7}In_{0.3}S_4$. The magnetic disorder there is present due to the competition of the ferromagnetic nearest neighbors interactions and the antiferromagnetic higher order neighbors interactions. This magnet has been already systematically studied some time ago [17], and its spin glass phase transition point $T = 16.7K$ is well established. Some of the measurements have been also performed on the metallic spin glasses $AgMn$ [18] and the results obtained were qualitatively quite similar. It indicates that presumably the qualitative physical phenomena observed, do not depend very much on the concrete realization of the spin glass system.

6.1 Aging

The phenomenon of *aging* in spin glasses is known already for many years [19]. It is not directly connected with the hierarchy of the spin-glass states, but it explicitly demonstrates the absence of true thermodynamic equilibrium in spin glasses.

The procedure of the experiments is in the following. The sample is cooled down into the spin-glass state in the presence of weak uniform magnetic field h. Then, at a constant temperature $T < T_c$ the sample is kept in this magnetic field during some waiting time t_w. Finally the magnetic field is switched off, and the measurements of the relaxation of the thermoremainent magnetization (TRM) is performed. The results of these measurements for different values of t_w is shown in Fig.12.

The first important result of these measurements is that the observed relaxation is extremely slow and non-exponential (note, that the typical values of t_w are well macroscopic: minutes, hours, days). More important, however, is that the relaxation appears to be *non-stationary*: the relaxation processes which take place in the system *after* switching off the field depend on the "lifetime" t_w of the system *before* the measurement was started. The spin glass is getting "stiffer" with the time: the bigger t_w is, the slower the relaxation goes on. Therefore, the results of the measurements depend on two time scales: the observation time t, and the time which has passed after the system came into the spin glass state, the "aging" time t_w. It is crucial that at all experimentally accessible time scales it has been observed no indication that the relaxation curves are reaching saturation at some limiting curve corresponding to $t_w = \infty$. Thus, at any experimentally accessible times such system remains out of the true thermal equilibrium.

Note that it is not the presence of the magnetic field, which is responsible for the observed phenomenon. The magnetic field here is just the instrument which makes possible to demonstrate it. One can also perform the "mirror" experiment: the system is cooled down into the spin glass state in the zero magnetic field, then it is kept at a constant temperature $T < T_c$ during some waiting time t_w, and finally the magnetic field is switched on and the relaxation of the magnetization is measured. Again, the results of the measurements essentially depend on t_w. Moreover, for any given value of t_w the curves obtained in these two types of the experiments appear to be symmetric: the sum of the values of the magnetizations obtained in these "mirror" experiments appears to be time independent constant (Fig.13).

6.2 Temperature cycles and the hierarchy of states

Now we consider two types of the experiments which are supposed to reveal the effects connected with the existence of the hierarchical tree of spin-glass states.

In the experiments of the first type, the sample in a weak magnetic field is cooled down into the spin-glass phase. Then, it is kept at a constant temperature $T < T_c$ during some waiting time t_{w_1}. After that the temperature is slightly changed down to $T' = (T - \Delta T)$ (where the value of ΔT is

small), and the sample is kept at this temperature during waiting time t_{w_3}. Then the temperature is changed up to the original value T again, and the sample is kept at this constant temperature during waiting time t_{w_2}. After that the magnetic field is switched off and the relaxation of the magnetization is measured. The results for different values of ΔT are shown in Fig.14.

The main result of these measurements is in the following. It is clear from the plots of Fig.14 that if the value of the temperature step ΔT is not too small, then all the relaxation curves appear to be *identical* to those in the usual aging experiments (Section 6.1) with the waiting time $t_w = t_{w_1} + t_{w_2}$. It means that for the processes of equilibration at the temperature T, the system remained effectively completely frozen during all the time period t_{w_3} when it was kept at the temperature $(T - \Delta T)$.

In the experiments of the second type, again, the sample in the presence of a weak magnetic field is cooled down into the spin-glass phase, and then it is kept at a constant temperature $T < T_c$ during waiting time t_{w_1}. Next, the sample is slightly heated up to the temperature $T' = (T + \Delta T)$, (where the value of ΔT is small) and after relatively short time interval it is cooled down again to the original temperature T. Then, it is kept at this constant temperature during waiting time t_{w_2}, and after that, the magnetic field is switched off and the relaxation of the magnetization is measured. The results for different values of ΔT are shown in Fig.15.

In this case one finds that if the value of the temperature step ΔT is not too small, then all the relaxation curves appear to be *identical* to those in the usual aging experiments (Section 6.1) with the waiting time $t_w = t_{w_3}$. It means that even slight heating is enough to wipe out all the aging which has been "achieved" at the temperature T during the time period before heating. In other words, after the slight heating jump the equilibration processes start all over again, while all the "pre-history" of the sample appears to be wiped out. (Note that the temperature $(T + \Delta T)$ is still well below T_c.)

Such quite asymmetric response of the system with respect to the considered temperature cycles of cooling and heating can be well explained in terms of the qualitative physical picture of the continuous hierarchy of the phase transitions and the tree-like structure of the spin-glass states.

The qualitative interpretation of the results described above is in the following. The process of thermal equilibration, as time goes on, can be imagined as the process of jumping over higher and higher energy barriers in the space of states. After some waiting time t_w the system covers certain part of the configurational space, which could be characterized by the maximum energy barriers of the order of $\Delta_{max} \simeq T \log(t_w/\tau)$ (here τ is characteristic microscopic time). It is assumed that any scale in the configurational space is characterized by certain typical value of the energy barriers (see also Section 5.4). Then the results of the experiments with the temperature cycles of cooling could be interpreted as follows. During the time period t_{w_1} when the system is kept at the temperature T, it covers certain finite part of the configurational space inside one of the valleys. After cooling down to the temperature $(T - \Delta T)$ this part of the configurational space is splitted into several smaller valleys separated by *infinite* energy barriers. Besides, the finite energy barriers separating the metastable states inside the valleys are getting higher, while some of these metastable states are splitted into many new ones. Then, during the time t_{w_3} the system is trying to cover these new states being locked by infinite barriers in a limited part of the configurational space. Therefore, whatever time has passed at the temperature $(T - \Delta T)$ the system can cover only those states, which are the descendants of the states already occupied at the temperature T, and not more. Note that this phenomenon of *ergodicity breaking* is just the consequence of the *phase transition* which occurred in the system due to cooling down from the temperature T to the temperature $(T - \Delta T)$. Then, after heating back to the original temperature T all these descendant states are merging together into their ancestors, and the process of thermal equilibration at the temperature T continues again, as if there was no time interval when the system spent at the temperature $(T - \Delta T)$.

In the experiments with the temperature cycles of heating the effects to be expected are different. After heating to the temperature $(T + \Delta T)$ the states occupied by the system during the time t_{w_1} at the temperature T would merge together into smaller number of their ancestor states. If the value of ΔT is not too small, such that $q(T + \Delta T) < q'$, where $q(T)$ is the selfoverlap of the states at the temperature T, and q' is the selfoverlap of the common ancestor of the states occupied during time interval t_{w_1}, then after heating all the occupied states would merge together into one common ancestor state. Within this limited part of the phase space this effectively corresponds to the paramagnetic phase transition. Therefore, all the thermal equilibration "achieved" at the temperature T will be wiped out, and after cooling back to the original temperature T the process of thermal equilibration

will start all over again.

In brief, the results of the considered experiments could be summarized as follows. If the spin-glass system is equilibrating at some temperature $T < T_c$, then any temporary heating would eliminate all the equilibration achieved, while any cooling for any time period, just postpones the equilibration processes at this temperature.

6.3 Temperature dependence of the energy barriers

The scheme of the above experiments can be slightly changed so that it would make possible to estimate the temperature dependence of the (finite) free energy barriers.

The experiments have been done on the metallic spin glasses $AgMn$ ($T_c = 10.4K$). The scheme of the experiments is in the following. First, the spin glass is aging in a weak magnetic field during the waiting time t_w at the temperature $(T - \Delta T)$. Then the sample is quickly heated up to the temperature T, and simultaneously the magnetic field is switched off. After that, the measurements of the relaxation of the magnetization is observed.

The results are shown in Fig.16. These plots clearly show, that if the value of ΔT is not too small, then the relaxation curves obtained are practically identical to those in the usual aging experiments at the same temperature T but with some other waiting time $t_w^{eff} < t_w$.

Assuming that the values of the *finite* energy barriers separating metastable states essentially depend on the temperature this phenomenon can also be easily explained in terms of the hierarchical structure of the spin-glass states. Since the free energy barriers at the temperature $(T - \Delta T)$ must be higher than corresponding barriers at the temperature T, the region of the phase space occupied by the system at the temperature $(T - \Delta T)$ is bounded by the barriers which are getting smaller at the temperature T. Correspondingly, the time needed to cover this part of the phase space at the temperature T is smaller than that at the temperature $(T - \Delta T)$. The crucial point is in the following. At the initial moment of the measurements the values of the temperature and the magnetic field in these two types of the experiments are the same, and if the value of t_w^{eff} is chosen correctly, then the long-time relaxation curves obtained appear to be identical. It means that the region of the phase space occupied by the system at the initial moment of the measurements in both cases must be the same. If the system is equilibrating at the temperature T this region can be characterized by the maximum value of the typical energy barriers:

$$\Delta(T; t_w^{eff}) \;=\; T \log(\frac{t_w^{eff}}{\tau}) \tag{6.1}$$

Correspondingly, if the equilibration takes place at the temperature $(T - \Delta T)$, the typical value of the maximum barriers is:

$$\Delta(T - \Delta T; t_w) \;=\; (T - \Delta T) \log(\frac{t_w}{\tau}) \tag{6.2}$$

Since the relaxation processes both after the aging at temperature T during the time t_w and after the aging at the temperature $(T - \Delta T)$ during the time t_w^{eff} are the same, the initial state of the system must also be the same. Therefore, one can conclude that $\Delta(T - \Delta T)$ and $\Delta(T)$ are the heights of *the same* barrier at different temperatures. Basing on this conclusion and using the experimental data of Fig.16, one can get the plot for the dependence of the value $\partial\Delta/\partial T$ from Δ at the given temperature. In Fig.17 the dependence of $\Delta(T - \Delta T)$ from $\Delta(T)$ is shown for $T = 9K,\ 9.5K$ and $10K$ at fixed value of the temperature jump $\Delta T = 20mK$. These plots demonstrate that within the experimental errors the dependencies obtained at different T coincide.

In Fig.18 the corresponding dependence of the value $\partial\Delta/\partial T$ from Δ is shown. Within the experimental errors the value of $\partial\Delta/\partial T$ depends only on Δ and it does not depend directly from the temperature. The dashed line in the Fig.18 is the power law fitting of the experimental data:

$$\frac{d\Delta}{dT} \;\simeq\; a\Delta^6\;; \qquad a \;=\; 2.9 \times 10^{-7} \tag{6.3}$$

Integrating this equation, one gets:

$$\Delta(T) \simeq [\frac{T-T^*}{T_c}]^{-1/5}\,; \quad T > T^* \tag{6.4}$$

The temperature T^* is the integration constant, which actually labels the concrete barrier. In other words, each barrier can be characterized by the critical temperature T^* at which this (finite at $T > T^*$) barrier becomes infinite. In this sense the critical temperature T_c can be interpreted just as the maximum possible value of T^*.

In conclusion, the experiments considered above clearly demonstrate the absence of the thermal equilibrium in the spin-glass phase at all experimentally accessible time scales. These experiments also demonstrate the existence of the whole spectrum of the free energy barriers up to infinite values, at any temperature below T_c. The results of the measurements show that the barriers heights strongly depend on the temperature and at *any* temperature $T < T_c$ certain barriers are getting infinite. This phenomenon clearly indicates on the presence of the ergodicity breaking phase transition at any temperature below T_c, which results in the continuous process of fragmentation of the phase space into smaller and smaller valleys with decrease of the temperature.

Part II. CRITICAL PHENOMENA AND QUENCHED DISORDER

7 Scaling Theory of the Critical Phenomena

7.1 The Ginzburg-Landau theory

We begin our study of the critical phenomena at the phase transitions of the second order with the mean-field approximation discussed in Introduction (Section 1.2). The starting point for further consideration is the mean-field expansion of the free energy in the vicinity of the critical point T_c, eq.(1.28), Fig.1:

$$f(\phi) \;=\; \frac{1}{2}\tau\phi^2 \;+\; \frac{1}{4}g\phi^4 \;-h\phi \tag{7.1}$$

where $\tau = (T - T_c)/T_c \ll 1$ is the reduced temperature, h is the external magnetic field. Here the "coupling constant" g is the parameter of the theory, and the order parameter $\phi = \langle\sigma_i\rangle$ is the average spin magnetization. The value of ϕ is determined from the condition of minimum of the free energy, $df/d\phi = 0$:

$$\tau\phi + g\phi^3 = h \tag{7.2}$$

and $d^2f/d\phi^2 > 0$.

In the absense of the external magnetic field ($h = 0$) at temperatures above T_c, ($\tau > 0$) the free energy has only one (trivial) minimum at $\phi = 0$. Below the critical point, $\tau < 0$, the free energy has two minima, and the corresponding solutions of the saddle-point equation (7.2) are:

$$\phi(\tau) \;=\; \pm\sqrt{\frac{|\tau|}{g}} \tag{7.3}$$

As $T \to T_c$ from below, $\phi(T) \to 0$.

As it has been already discussed in the Introduction, this very simple mean-field theory demonstrate on a qualitative level the fundamental phenomenon called the spontaneous symmetry breaking. At the critical temperature $T = T_c$ the phase transition of the second order occurs, such that in the low temperature region $T < T_c$ the symmetry with respect to the global change of the signs of the spins is broken, and the *two* (instead of one) ground states appear. These two states differ by the sign of the average spin magnetization, and they are separated by the macroscopic barrier of the free energy.

In a small nonzero magnetic field ($h \ll 1$) the qualitative shape of the free energy is shown in Fig.1b. In this case the saddle-point equation (7.2) always has nonzero solution for the order parameter ϕ at all temperatures. In particular, in the low-temperature region ($\tau < 0$) we find:

$$\phi \simeq \begin{cases} \sqrt{\frac{|\tau|}{g}} + \frac{h}{2\tau} & \text{if } h \ll h_c(\tau) \\ (\frac{h}{g})^{1/3} & \text{if } h \gg h_c(\tau) \end{cases} \tag{7.4}$$

where

$$h_c(\tau) = \frac{1}{\sqrt{g}}|\tau|^{3/2} \tag{7.5}$$

whereas in the high-temperature region ($\tau > 0$):

$$\phi \simeq \begin{cases} \frac{h}{\tau} & \text{if } h \ll h_c(\tau) \\ (\frac{h}{g})^{1/3} & \text{if } h \gg h_c(\tau) \end{cases} \tag{7.6}$$

Thus, at $h \neq 0$ the phase transition is "smoothed out" in the temperature interval $|\tau| \sim h^{2/3}$ [eq.(7.5)] around T_c.

The physical quantity, which describes the reaction of the system on the infinitely small magnetic field is called susceptibility. It is defined as follows:

$$\chi \;=\; \frac{\partial \phi}{\partial h}\mid_{h=0} \tag{7.7}$$

According to eqs.(7.4)-(7.6) one finds that near the critical point the susceptibility becomes divergent:

$$\chi \simeq \begin{cases} \tau^{-1} & \text{at } T > T_c \\ \frac{1}{2}|\tau|^{-1} & \text{at } T < T_c \end{cases} \tag{7.8}$$

For the so called nonlinear susceptibility $\chi(h) = \partial\phi/\partial h$ in the close vicinity of the critical point ($|\tau|^{3/2} \ll h\sqrt{g}$), we get:

$$\chi(h) \simeq h^{-2/3} \tag{7.9}$$

The other basic physical quantity is the specific heat, which is defined as follows:

$$C \;=\; -T\frac{\partial^2 f}{\partial T^2} \tag{7.10}$$

For the specific heat near the critical point (in the zero magnetic field), according to the eq.(7.3)-(7.1) we obtain:

$$C \simeq \begin{cases} const = \frac{1}{2g} & \text{at } T > T_c \\ 0 & \text{at } T < T_c \end{cases} \tag{7.11}$$

Of course, all the above results which were obtained in terms of very primitive mean-field approximation cannot pretend to be reliable. Nevertheless, on a qualitative level they demonstrate very important physical phenomenon: near the point of the second-order phase transition at least some of the physical quantities become singular (or non-analytic). Now let us consider one simple and natural improvement of the mean-field theory considered above.

The apparent defect of the mean-field approximation given above is that it does not take into account correlations among spins. This could be easily amended if we are interested in the studies of only *large-scale* phenomena which will be shown to be responsible for the leading singularities in the thermodynamical functions. In this case the order parameters ϕ_i are almost spatially homogeneous, and they can be represented as slowly varying (with small gradients) functions of the continuous space coordinates. Then, the interaction term in the exact lattice Hamiltonian (1.14) can be approximated as follows:

$$\frac{1}{2}\sum_{<i,j>}\phi_i\phi_j \;\rightarrow \frac{1}{2}\int d^D x[\phi^2(x) + (\nabla\phi(x))^2] \tag{7.12}$$

Correspondingly, the Hamiltonian in which only small spatial fluctuations of the order parameter are taken into account can be written as follows:

$$H = \int d^D x[\frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}\tau\phi^2(x) + \frac{1}{4}g\phi^4(x) - h\phi(x)] \tag{7.13}$$

The theory which is based on the above Hamiltonian is called the Ginzburg-Landau approach. In fact the Ginzburg-Landau Hamiltonian is nothing but the first few terms of the expansion in powers of ϕ and $(\nabla\phi)$. In the vicinity of the (second-order) phase transition point, where the order parameter is small and the leading contributions come from large-scale fluctuations, such an approach looks quite natural.

Consider the contributions caused by small fluctuations at the background of the homogeneous order parameter $\phi_0 = \sqrt{|\tau|/g}$:

$$\phi(x) = \phi_0 + \varphi(x) \tag{7.14}$$

where $\varphi(x) \ll \phi_0$.

For simplicity let us consider the case of the zero magnetic field. Then the expansion of the Hamiltonian (7.13) to the second order in φ yields:

$$H = H_0 + \int d^D x [\frac{1}{2}(\nabla\varphi(x))^2 + |\tau|\varphi^2(x)] \tag{7.15}$$

In terms of the Fourier representation

$$\varphi(x) = \int \frac{d^D k}{(2\pi)^D}\varphi(k)\exp(-ikx) \tag{7.16}$$

one gets:

$$H = \frac{1}{2}\int \frac{d^D k}{(2\pi)^D}(k^2 + 2|\tau|) \mid \varphi(k) \mid^2 + H_0 \tag{7.17}$$

Therefore, for the correlation function

$$G_0(k) \equiv \langle \mid \varphi(k) \mid^2 \rangle = \frac{\int D\varphi(k) \mid \varphi(k) \mid^2 \exp(-H[\varphi])}{\int D\varphi(k)\exp(-H[\varphi])} \tag{7.18}$$

one obtains the following result:

$$G_0(k) = \frac{1}{k^2 + 2|\tau|} \tag{7.19}$$

Besides, it is obvious that

$$\langle\varphi(k)\varphi(k')\rangle = G_0(k)\delta(k + k') \tag{7.20}$$

Therefore, for the spatial correlation function

$$\begin{aligned} &G_0(x) = \langle\langle\phi(0)\phi(x)\rangle\rangle \equiv \langle\phi(0)\phi(x)\rangle - \langle\phi(0)\rangle\langle\phi(x)\rangle = \\ &= \langle\varphi(0)\varphi(x)\rangle = \textstyle\int \frac{d^D k}{(2\pi)^D}\langle \mid \varphi(k) \mid^2\rangle \exp(ikx) \end{aligned} \tag{7.21}$$

we obtain:

$$G_0(x) \sim \begin{cases} \mid x \mid^{-(D-2)} & \text{for } \mid x \mid \ll R_c(\tau) = \frac{1}{\sqrt{2|\tau|}}; \quad (a) \\ \exp(- \mid x \mid /R_c) & \text{for } \mid x \mid \gg R_c(\tau); \qquad\qquad (b) \end{cases} \tag{7.22}$$

Here the quantity

$$R_c(\tau) \sim |\tau|^{-1/2} \tag{7.23}$$

is called the correlation length.

Thus, the situation near T_c ($|\tau| \ll 1$) looks as follows. At scales much larger than the correlation length $R_c(\tau) \gg 1$ the fluctuations of the field $\phi(x)$ around its equilibrium value ϕ_0 ($\phi_0 = 0$ at $T > T_c$, and $\phi_0 = \sqrt{|\tau|/g}$ at $T < T_c$) become effectively independent (their correlations decay exponentially, eq.(7.22(b)). On the other hand, at scales much smaller than $R_c(\tau)$, in the so called fluctuation region, the fluctuations of the order parameter are strongly correlated, and their correlation functions exibit weak power-law decay, eq.(7.22(a)). Therefore, inside the fluctuation region at scales $\ll R_c(\tau)$ the gradient, or the fluctuation term of the Hamiltonian (7.13) becomes crucial for the theory. In the critical point the fluctuation region becomes infinite.

Let us estimate to what extent the above simple considerations are correct. The expansion (7.15) could be used and the result (7.22) is justified only if the characteristic value of the fluctuations φ are small in comparison with the equilibrium value of the order parameter ϕ_0. Since the correlation length R_c is the only relevant spatial scale which exists in the system near the phase transition point, the characteristic value of the fluctuations of the order parameter could be estimated as follows:

$$\overline{\varphi^2} \equiv \frac{1}{R_c^D}\int_{|x|<R_c} d^D x\langle\varphi(0)\varphi(x)\rangle \;\sim\; R_c^{-(D-2)} \tag{7.24}$$

The above simple mean-field estimates for the critical behavior are grounded only if the value of $\overline{\varphi^2}$ is much smaller than the corresponding value of the equilibrium order parameter ϕ_0^2:

$$R_c^{-D+2} \ll \frac{|\tau|}{g} \tag{7.25}$$

Using (7.23) we find that this condition is satisfied if:

$$g|\tau|^{\frac{D-4}{2}} \ll 1 \tag{7.26}$$

Therefore, if the dimension of the system is bigger than 4, near the phase transition point, $\tau \to 0$, the condition (7.26) is always satisfied. On the other hand, at dimensions $D < 4$ this condition is always violated near the critical point.

Thus, these simple estimates reveal the following quite important points:

1) If the dimension of the considered system is bigger than 4, then its critical behavior in the vicinity of the second order phase transition is successfully described by the mean-field theory.

2) If the dimension of the system is less than 4, then, according to eq.(7.26), the mean-field approach gives correct results only in the range of temperatures not too close to T_c:

$$\tau \gg \tau_*(D, g) \equiv g^{\frac{2}{4-D}} , \qquad (\tau \ll 1) \tag{7.27}$$

(here it is assumed that $g \ll 1$, otherwise there would be no mean-field critical region at all). In the close vicinity of T_c, $|\tau| \ll \tau_*$, the other (non-Gaussian) type of the critical behavior can be expected to occur.

7.2 Critical Exponents

In general, it is believed that critical behavior of the physical quantities near the phase transition point can be described in terms of the so-called *critical exponents*. In particular, for the quantities considered above, the critical exponents are defined as follows:

Correlation length	$R_c \sim \mid \tau \mid^{-\nu}$	at $h \ll h_c(\tau)$
	$R_c \sim h^{-\mu}$	at $h \gg h_c(\tau)$
Order parameter:	$\phi_0 \sim \|\tau\|^{\beta}$	at $h \ll h_c(\tau)$; $\tau < 0$
	$\phi_0 \sim h^{1/\delta}$	at $h \gg h_c(\tau)$
Specific heat:	$C \sim \mid \tau \mid^{-\alpha}$	at $h \ll h_c(\tau)$
Susceptibility:	$\chi \sim \mid \tau \mid^{-\gamma}$	at $h \ll h_c(\tau)$
	$\chi \sim h^{1/\delta - 1}$	at $h \gg h_c(\tau)$
Correlation function	$G(x) \sim \mid x \mid^{-D+2-\eta}$	at $\mid x \mid \ll R_c$

(7.28)

where the value of the critical field is $h_c(\tau) \sim \mid \tau \mid^{\nu/\mu}$ (this estimate follows from the comparison of the correlation lengths in small and in large fields).

In fact, not all the critical exponents listed in eq.(7.28) are independent. One can easily derive (see below) the following relations among them:

$$\alpha = 2 - D\nu \tag{7.29}$$

$$\delta = \frac{D + 2 - \eta}{D - 2 + \eta} \tag{7.30}$$

$$\gamma = (2 - \eta)\nu \tag{7.31}$$

$$2\beta = 2 - \gamma - \alpha \tag{7.32}$$

$$\mu = \frac{2}{D + 2 - \eta} \tag{7.33}$$

For 7 exponents there are exist 5 equations, which means that only two exponents are independent. In other words, to find all the critical exponents one needs to calculate only two of them.

In particular, the Ginzburg-Landau mean-field theory considered above, gives: $\nu = 1/2$ and $\eta = 0$ (see eqs.(7.22)-(7.23)). Using eqs.(7.29)-(7.33) one easily finds the rest of the exponents: $\alpha = -(D-4)/2,\ \delta = \frac{D+2}{D-2},\ \gamma = 1,\ \beta = (D-2)/4,\ \mu = 1/3$. These critical exponents fully describe the critical behavior of any scalar field D-dimensional system with $D \geq 4$.

Let us now derive the relations (7.29)-(7.33). According to the definition of specific heat:

$$C = -T\frac{\partial^2 f}{\partial T^2} \tag{7.34}$$

one gets:

$$C = \frac{1}{V}\int d^D x \int d^D x' [\langle \phi^2(x)\phi^2(x')\rangle - \langle \phi^2(x)\rangle\langle \phi^2(x')\rangle] \sim \frac{1}{R_c^D}\langle \Phi\rangle^2 \tag{7.35}$$

where

$$\Phi = \int_{|x|<R_c} d^D x \phi^2(x) \tag{7.36}$$

According to eq.(7.13), the equilibrium energy density of the system (at scales bigger than R_c) is proportional to $|\tau|\Phi$. Thus, the equilibrium value of $\langle\Phi\rangle$ is defined by the condition $|\tau|\langle\Phi\rangle \sim T$ ($T \simeq T_c = 1$ in our case). Therefore, from eq.(7.35) one gets:

$$C \sim R_c^{-D}|\tau|^{-2} \sim |\tau|^{D\nu-2} \tag{7.37}$$

On the other hand, according to the definition of the critical exponent α, $C \sim |\tau|^{-\alpha}$, and one obtains the eq.(7.29).

Using the definitions of the susceptibility, as well as the critical exponents of the correlation function η and that of the correlation length ν, eq.(7.28), one obtains:

$$\begin{aligned} &\chi = \tfrac{\partial\langle\phi\rangle}{\partial h}|_{h=0} = \textstyle\int d^D x \langle\langle\phi(0)\phi(x)\rangle\rangle \sim \\ &R_c^D R_c^{2-D-\eta} \sim |\tau|^{-\nu(2-\eta)} \end{aligned} \tag{7.38}$$

On the other hand: $\chi \sim |\tau|^{-\gamma}$, which provides the eq.(7.31).

The value of the susceptibility, eq(7.38), can be estimated in the other way:

$$\chi \sim R_c^D \phi_0^2 \sim |\tau|^{-D\nu+2\beta} \tag{7.39}$$

This yields: $\gamma = D\nu - 2\beta$. Using eq.(7.29), one gets eq.(7.32).

Now let us define the value of the order parameter in the region, which is less than the correlation length:

$$\psi \equiv \int_{|x|<R_c} d^D x \phi(x) \tag{7.40}$$

The characteristic value of the field ψ is:

$$\begin{aligned} &\psi_c \equiv \sqrt{\langle\psi^2\rangle} \sim \\ &\sim (R_c^D \textstyle\int_{|x|<R_c} d^D x \langle\phi(0)\phi(x)\rangle)^{1/2} \sim R_c^{\frac{D+2-\eta}{2}} \end{aligned} \tag{7.41}$$

The critical value of the external field $h_c(\tau)$ is defined by the condition:

$$\psi_c h_c \sim T\ (=1) \tag{7.42}$$

Therefore, at this value of the field:

$$R_c(h) \sim h^{-\frac{2}{D+2-\eta}} \tag{7.43}$$

which yields eq.(7.33).

On the other hand: $\psi_c \sim \phi_0 R_c$. Using the condition (7.42), the result (7.43) and the definition: $\phi_0 \sim h^{1/\delta}$, one gets:

$$\psi_c \sim \frac{1}{h} \sim h^{\frac{1}{\delta}} h^{-\frac{2D}{D+2-\eta}} \tag{7.44}$$

Simple algebra gives the relation (7.30).

In actual calculations one usually obtains the critical exponent of the correlation length ν, and that of the correlation function η, while the rest of the exponents are derived from the relations (7.29)-(7.33) automatically.

7.3 Scaling

The concepts of the critical exponents and the correlation length are crucial for the theory of the second-order phase transitions. In the scaling theory of the critical phenomena it is implied that R_c is the only relevant spatial scale which exists in the system near T_c. As we have seen in the GL mean-field approach discussed above, at scales smaller than R_c all the spatial correlations are power-like, which means that at scales much smaller than the correlation length everything must be scale-invariant. On the other hand, in the phase transition point the correlation length is infinite. Therefore, the properties of the system at scales smaller then R_c must be equivalent to those of the whole system at the phase transition point.

The other important consequence of scale invariance is that the microscopic details of a system (lattice structure, etc.) should not be expected to affect the critical behavior. What may appear to be relevant for the critical properties of a system are only its "global" characteristics, such as space dimensionality, topology of the order parameter, etc. All the above arguments make a basis for the so-called *scaling hypothesis*, which claims that the macroscopic properties of a system at the critical point do not change after a global change of the spatial scale.

Let us consider, in brief, what the immediate general consequences of such a statement would be. Let the Hamiltonian of a system be the following:

$$H = \int d^D x [\frac{1}{2}(\nabla\phi(x))^2 + \sum_{n=1} h_n \phi^n(x)] \tag{7.45}$$

Here the parameters h_n describe a concrete system under consideration. In particular: $h_1 \equiv -h$ is the external field; $h_2 \equiv \tau$ is the "mass" in the Ginzburg-Landau theory; $h_4 \equiv \frac{1}{4}g$; and the rest of the parameters could describe some other types of interactions. After the scale transformation:

$$x \;\rightarrow\; \lambda x \qquad (\lambda > 1) \tag{7.46}$$

one gets:

$$\begin{aligned} &\tfrac{1}{2}\textstyle\int d^D x(\nabla\phi(x))^2 \;\rightarrow\; \tfrac{1}{2}\lambda^{D-2}\int d^D x(\nabla\phi(\lambda x))^2 \\ &h_n \textstyle\int d^D x\phi^n(x) \;\rightarrow \lambda^D \int d^D x\phi^n(\lambda x) \end{aligned} \tag{7.47}$$

To leave the gradient term of the Hamiltonian (which is responsible for the scaling of the correlation functions) unchanged, one has to rescale the fields:

$$\phi(\lambda x) \;\rightarrow\; \lambda^{-\Delta_\phi}\phi(x) \tag{7.48}$$

with

$$\Delta_\phi = \frac{D-2}{2} \tag{7.49}$$

The *scale dimensions* Δ_ϕ defines the critical exponent of the correlation function:

$$G(x) = \langle\phi(0)\phi(x)\rangle \sim |x|^{-2\Delta_\phi} \tag{7.50}$$

To leave the Hamiltonian (7.45) unchanged after these transformations one must also rescale the parameters h_n:

$$h_n \;\to\; \lambda^{-\Delta_n} h_n \tag{7.51}$$

where

$$\Delta_n \;= \frac{1}{2}(2-n)D + n \tag{7.52}$$

The quantities Δ_n are called the *scale dimensions* of the corresponding parameters h_n. In particular:

$$\Delta_1 \equiv \Delta_h = \frac{1}{2}D + 1 \tag{7.53}$$

$$\Delta_2 \equiv \Delta_\tau = 2 \tag{7.54}$$

$$\Delta_4 \equiv \Delta_g = 4 - D \tag{7.55}$$

Correspondingly, the rescaled parameters h_λ, τ_λ and g_λ of the Ginzburg-Landau Hamiltonian are:

$$h_\lambda \;=\; \lambda^{\Delta_h} h \tag{7.56}$$

$$\tau_\lambda \;=\; \lambda^{\Delta_\tau} \tau \tag{7.57}$$

$$g_\lambda \;=\; \lambda^{\Delta_g} g \tag{7.58}$$

These equations demonstrate the following points:

1) If the initial value of the "mass" τ is non-zero, then the scale transformations make the value of the rescaled τ_λ to grow, and at the scale

$$\lambda_c \equiv R_c = |\tau|^{-\frac{1}{\Delta_\tau}} \tag{7.59}$$

the value of τ_λ becomes of the order of 1. This indicates that at $\lambda > R_c$ we are getting out of the scaling region, and the value R_c must be called the correlation length. Moreover, according to eq.(7.59) for the critical exponent of the correlation length we find:

$$\nu \;=\; \frac{1}{\Delta_\tau} \tag{7.60}$$

2) The value (and the critical exponent) of the critical field $h_c(\tau)$ can be obtained from the eqs.(7.53),(7.56) along the same lines:

$$\begin{aligned} & h_\lambda|_{\lambda=R_c} = R_c^{\Delta_h} h_c \;\sim 1 \quad \Rightarrow \\ & \Rightarrow \quad h_c \sim R_c^{-\Delta_h} \;\sim\; |\tau|^{\frac{\Delta_h}{\Delta_\tau}} \end{aligned} \tag{7.61}$$

3) If the dimension of the system is greater than 4, then according to eqs.(7.55) and (7.58), $\Delta_g < 0$, and the rescaled value of the parameter g_λ tends to zero at infinite scales. Therefore, the theory becomes asymptotically Gaussian in this case. That is why the systems with dimensions $D > 4$ are described correctly by the Ginzburg-Landau theory.

On the other hand, at dimension $D < 4$, $\Delta_g > 0$, and the rescaled value of g_λ grows as the scale increases. In this case the situation becomes highly nontrivial because the asymptotic (infinite scale) theory becomes non-Gaussian. Nevertheless, if the dimension D is formally taken to be close to 4, such that the value of $\epsilon = 4 - D$ is treated as the small parameter, then the deviation from the Gaussian theory is also small in ϵ, and this allows us to treat such systems in terms of the perturbation theory (see next Section). In the lucky case, if for some reasons the series in ϵ would appears to be "good" and quickly converging, then one could hope to get the critical exponents close to the real ones if we set $\epsilon = 1$ in the final results.

It is a miracle, but although the actual series in ϵ can by no means be considered as "good" (it is not even converging), the results for the critical exponents given by the first three terms of the series at $\epsilon = 1$ ($D = 3$) appear to be very close to the real ones.

7.4 Renormalization-group approach and ϵ-expansion

Let us assume that at large scales the asymptotic theory is described by the Hamiltonian (7.13) (for simplicity the external field h is taken to be zero):

$$H = \int d^D x [\frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}\tau\phi^2(x) + \frac{1}{4}g\phi^4(x)] \tag{7.62}$$

where the field $\phi(x)$ is supposed to be slow-varying in space, such that the Fourier-transformed field $\phi(k)$:

$$\phi(x) = \int_{|k|<k_0} \frac{d^D k}{(2\pi)^D}\phi(k)\exp(ikx) \tag{7.63}$$

has only long-wave components: $\mid k \mid < k_0 \ll 1$. The parameters of the Hamiltonian are also assumed to be small: $|\tau| \ll 1; \quad g \ll 1$. Correspondingly, the Fourier-transformed Hamiltonian is:

$$\begin{aligned} H_{k_0} = \tfrac{1}{2}\textstyle\int_{|k|<k_0} \frac{d^D k}{(2\pi)^D} k^2 \mid \phi(k) \mid^2 + \tfrac{1}{2}\tau \int_{|k|<k_0} \frac{d^D k}{(2\pi)^D} \mid \phi(k) \mid^2 + \\ + \tfrac{1}{4} g \int_{|k|<k_0} \frac{d^D k_1 d^D k_2 d^D k_3 d^D k_4}{(2\pi)^{4D}} \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\delta(k_1+k_2+k_3+k_4) \end{aligned} \tag{7.64}$$

In the most general terms the problem is to calculate the partition function:

$$Z = (\prod_{k=0}^{k_0} \int d\phi(k)) \exp\{-H_{k_0}(\phi)\} \tag{7.65}$$

and the corresponding free energy: $F = -\ln(Z)$.

The idea of the renormalization-group (RG) approach is described below.

In the **first step** one integrates only over the components of the field $\phi(k)$ in the limited wave band $\lambda k_0 < k < k_0$, where $\lambda \ll 1$. In the result we get a new Hamiltonian which depends on the new cutoff λk_0:

$$\exp\{-\tilde{H}_{\lambda k_0}[\phi]\} \equiv (\prod_{k=\lambda k_0}^{k_0} \int d\phi(k)) \exp(-H_{k_0}[\phi]) \tag{7.66}$$

It is expected that under certain conditions the new Hamiltonian $\tilde{H}_{\lambda k_0}[\phi]$ would have the structure similar to the original one, given by eq.(7.64):

$$\begin{aligned} \tilde{H}_{\lambda k_0} = \tfrac{1}{2}\tilde{a}(\lambda) \textstyle\int_{|k|<\lambda k_0} \frac{d^D k}{(2\pi)^D} k^2 \mid \phi(k) \mid^2 + \tfrac{1}{2}\tilde{\tau}(\lambda) \int_{|k|<\lambda k_0} \frac{d^D k}{(2\pi)^D} \mid \phi(k) \mid^2 + \\ + \tfrac{1}{4}\tilde{g}(\lambda) \int_{|k|<\lambda k_0} \frac{d^D k_1 d^D k_2 d^D k_3 d^D k_4}{(2\pi)^{4D}} \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\delta(k_1+k_2+k_3+k_4) + (...) \end{aligned} \tag{7.67}$$

All additional terms which could appear in $\tilde{H}_{\lambda k_0}[\phi]$ after the integration (7.66) (denoted by "(...)") will be shown to be irrelevant for $\tau \ll 1, \ \ g \ll 1, \ \ \lambda \ll 1$, and $\epsilon = (4-D) \ll 1$. In fact, the leading terms in (7.67) will be shown to be large in the parameter $\xi \equiv \ln(1/\lambda) \gg 1$, conditioned that $\epsilon\ln(1/\lambda) \ll 1$.

In the **second step** one makes the inverse scaling transformation (see Section 7.3) with the aim of restoring the original cutoff scale k_0:

$$\begin{aligned} k \ &\rightarrow \ \lambda k \\ \phi(\lambda k) \ &\rightarrow \ \theta(\lambda)\phi(k) \end{aligned} \tag{7.68}$$

The parameter $\theta(\lambda)$ should be chosen such that the coefficient of the $k^2 \mid \phi(k) \mid^2$ term remains the same as in the original Hamiltonian (7.64):

$$\theta = \lambda^{-\frac{D+2}{2}} (\tilde{a}(\lambda))^{-1/2} \tag{7.69}$$

The two steps given above compose the so-called *renormalization transformation*. The renormalized Hamiltonian is:

$$H_{k_0}^{(R)} = \frac{1}{2}\int_{|k|<k_0} \frac{d^D k}{(2\pi)^D} k^2 \mid \phi(k) \mid^2 + \frac{1}{2}\tau^{(R)}(\lambda) \int_{|k|<k_0} \frac{d^D k}{(2\pi)^D} \mid \phi(k) \mid^2 +$$
$$+\frac{1}{4} g^{(R)}(\lambda) \int_{|k|<k_0} \frac{d^D k_1 d^D k_2 d^D k_3 d^D k_4}{(2\pi)^{4D}} \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\delta(k_1+k_2+k_3+k_4) \tag{7.70}$$

This Hamiltonian depends on the original cutoff k_0 whereas its parameters are renormalized:

$$\tau^{(R)}(\lambda) = \lambda^{-2}\tilde{a}(\lambda)^{-1}\tilde{\tau}(\lambda) \tag{7.71}$$

$$g^{(R)}(\lambda) = \lambda^{-(4-D)}\tilde{a}(\lambda)^{-2}\tilde{g}(\lambda) \tag{7.72}$$

The above the RG transformation must be applied (infinitely) many times, and then the problem is to study the limiting properties of the renormalized Hamiltonian, which is expected to describe the asymptotic (infinite scale) properties of the system. In particular, it is hoped that the asymptotic Hamiltonian would arrive at some fixed point Hamiltonian H^* which would be invariant with respect to the above RG transformation. The hypothesis about the existence of the fixed point (non-Gaussian) Hamiltonian H^* which would be invariant with respect to the scale transformations in the critical point is nothing but a more "mathematical" formulation of the scaling hypothesis discussed in the Section 7.3.

Let us consider the RG procedure in some more detail. To get the RG eqs.(7.71)-(7.72) in explicit form one has to obtain the parameters $\tilde{a}(\lambda), \tilde{\tau}(\lambda), \tilde{g}(\lambda)$ by integrating "fast" degrees of freedom in eq.(7.66). Let us separate the "fast" fields (with $\lambda k_0 < |k| < k_0$) and the "slow" fields (with $|k| < \lambda k_0$) explicitly:

$$\phi(x) \;=\; \tilde{\phi}(x) \;+\; \varphi(x)\,;$$
$$\tilde{\phi}(x) = \int_{|k|<\lambda k_0} \frac{d^D k}{(2\pi)^D} \tilde{\phi}(k)\exp(ikx); \qquad \varphi(x) = \int_{\lambda k_0<|k|<k_0} \frac{d^D k}{(2\pi)^D} \varphi(k)\exp(ikx) \tag{7.73}$$

Then the Hamiltonian (7.64) can be represented as follows:

$$H_{k_0}[\tilde{\phi},\varphi] = H_{\lambda k_0}[\tilde{\phi}] \;+\; \frac{1}{2}\int_{\lambda k_0<|k|<k_0} \frac{d^D k}{(2\pi)^D} G_0^{-1}(k) \mid \varphi(k) \mid^2 \;+ V[\tilde{\phi},\varphi] \tag{7.74}$$

where

$$G_0(k) \;=\; k^{-2} \tag{7.75}$$

and

$$V[\tilde{\phi},\varphi] = \frac{1}{2}\tau \int_{\lambda k_0<|k|<k_0} \frac{d^D k}{(2\pi)^D} \mid \varphi(k) \mid^2 \;+$$
$$+\frac{3}{2} g \int \frac{d^D k_1 d^D k_2 d^D k_3 d^D k_4}{(2\pi)^{4D}} \tilde{\phi}(k_1)\tilde{\phi}(k_2)\varphi(k_3)\varphi(k_4)\delta(k_1+k_2+k_3+k_4) +$$
$$+ g \int \frac{d^D k_1 d^D k_2 d^D k_3 d^D k_4}{(2\pi)^{4D}} \tilde{\phi}(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\delta(k_1+k_2+k_3+k_4) + \tag{7.76}$$
$$+ g \int \frac{d^D k_1 d^D k_2 d^D k_3 d^D k_4}{(2\pi)^{4D}} \tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\varphi(k_4)\delta(k_1+k_2+k_3+k_4) +$$
$$+\frac{1}{4} g \int \frac{d^D k_1 d^D k_2 d^D k_3 d^D k_4}{(2\pi)^{4D}} \varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\delta(k_1+k_2+k_3+k_4)$$

In standard diagram notations the interaction term $V[\tilde{\phi},\varphi]$ is shown in Fig.19, where the wavy lines represent the "slow" fields $\tilde{\phi}$, the straight lines represent the "fast" fields φ, the solid circle represents the "mass" τ, the open circle represents the interaction vertex g, and at each vertex the sum of entering "impulses" k is zero.

Then, the integration over the φ's, eq.(7.66), yields:

$$\exp\{-\tilde{H}_{\lambda k_0}[\tilde{\phi}]\} \;=\; \exp\{-H_{\lambda k_0}[\tilde{\phi}]\}\langle\exp\{-V[\tilde{\phi},\varphi]\}\rangle \tag{7.77}$$

where the averaging $\langle(...)\rangle$ is performed as follows:

$$\langle(...)\rangle \equiv (\prod_{k=\lambda k_0}^{k=k_0}\int d\varphi(k))\exp\{-\frac{1}{2}\int_{\lambda k_0<|k|<k_0}\frac{d^Dk}{(2\pi)^D}G_0^{-1}(k)\mid\varphi(k)\mid^2\}(...) \tag{7.78}$$

Standard perturbation expansion in V gives:

$$\tilde{H}_{\lambda k_0}[\tilde{\phi}] \;=\; H_{\lambda k_0}[\tilde{\phi}] \;+\; \langle V\rangle \;-\; \frac{1}{2}[\langle V^2\rangle - \langle V\rangle^2] \tag{7.79}$$

In terms of the diagrams, Fig.19, the averaging $\langle ...\rangle$ is just the pairing of the straight lines. The non-zero contribution to $\langle V\rangle$ is shown in Fig.20, where each closed loop is:

$$\int_{\lambda k_0<|k|<k_0}\frac{d^Dk}{(2\pi)^D}G_0(k) \;=\; \frac{S_D}{(2\pi)^D(D-2)}k_0^{(D-2)}(1-\lambda^{(D-2)}) \tag{7.80}$$

(here S_D is the surface area of a unite D-dimensional sphere).

In what follows we are going to study the limit case of the small cutoff k_0 (large spatial scales). Besides, at each RG step the rescaling parameter λ will also be assumed to be small, such that in all the integrations over the "internal" k's ($\lambda k_0 < |k| < k_0$) the "external" k's ($|k| < \lambda k_0$) could be considered as negligibly small.

The result for the first order perturbation expansion $\langle V\rangle$ consists of three contributions. The diagrams (a) and (c) in Fig.20 produce only irrelevant constants (they do not depend on $\tilde{\phi}$). The diagram (b) is proportional to $|\tilde{\phi}(k)|^2$ and gives the contribution to the mass term, but since this contribution is proportional to $k_0^{(D-2)}$, in the asymptotic region $k_0 \to 0$ it could be ignored as well. In fact we are going to look for the contributions, which: (1) do not depend on the value of the cutoff k_0; and (2) are large in the RG parameter $\xi \equiv \ln(1/\lambda) \gg 1$.

Consider the second-order perturbation contribution $\langle\langle V^2\rangle\rangle \equiv \langle V^2\rangle - \langle V\rangle^2$, Fig.21. Here the diagrams (a), (c) and (i) give irrelevant constant. The diagrams (d), (g) and (h) are proportional to the positive power of the cutoff k_0 and therefore their contribution is small.

The relevant diagrams are (b), (e) and (f). The diagram (e) is proportional to:

$$\begin{array}{l}\int_{|k|<\lambda k_0}\frac{d^Dk}{(2\pi)^D}|\tilde{\phi}(k)|^2\int_{\lambda k_0<|k_{1,2}|<k_0}d^Dk_1d^Dk_2G_0(k_1)G_0(k_2)G_0(k+k_1+k_2) \;= \\ \int_{|k|<\lambda k_0}\frac{d^Dk}{(2\pi)^D}|\tilde{\phi}(k)|^2\int_{\lambda k_0<|k_{1,2}|<k_0}\frac{d^Dk_1d^Dk_2}{k_1^2k_2^2(k+k_1+k_2)^2}\end{array} \tag{7.81}$$

since $k \ll k_{1,2}$ the leading contribution in (7.81) is given by the first terms of the expansion in $k/k_{1,2}$:

$$\begin{array}{l}g^2\int_{|k|<\lambda k_0}\frac{d^Dk}{(2\pi)^D}|\tilde{\phi}(k)|^2\int_{\lambda k_0<|k_{1,2}|<k_0}\frac{d^Dk_1d^Dk_2}{k_1^2k_2^2(k_1+k_2)^2} \;+ \\ +3g^2\int_{|k|<\lambda k_0}\frac{d^Dk}{(2\pi)^D}|\tilde{\phi}(k)|^2k^2\int_{\lambda k_0<|k_{1,2}|<k_0}\frac{d^Dk_1d^Dk_2}{k_1^2k_2^2(k_1+k_2)^4}\end{array} \tag{7.82}$$

The first contribution in (7.82) is of the order of $k_0^{(D-2)}$ is therefore irrelevant. As for the second contribution, it could be easily checked that at dimension $D = 4-\epsilon$, where $\epsilon \ll 1$, the integration over k_1 and k_2 does yield the factor proportional to $\ln(1/\lambda) \gg 1$ independent of the cutoff k_0. Therefore this diagram gives finite contribution of the order of $g^2\ln(1/\lambda)$ into $\tilde{a}$, eq.(7.67). However, as will be demonstrated below, the renormalized fixed-point value of g appears to be of the order of ϵ. It means that the diagram in Fig.3(e) dives the contribution of the order of $\epsilon^2\ln(1/\lambda)$ in $\tilde{a}$ (which provides the correction of the order of ϵ^2 into the critical exponents). Therefore, until we study only the first-order in ϵ corrections the contribution of the diagram (e) should not be taken into account:

$$\tilde{a} = 1 + O(g^2)\xi \tag{7.83}$$

where $\xi \equiv \ln(1/\lambda)$.

The diagram (b) of the Fig.21 gives the following contribution:

$$\begin{aligned} &\tfrac{3}{2}g\tau\int_{\lambda k_0<|k|<k_0}\tfrac{d^Dk}{(2\pi)^D}\tfrac{1}{k^4}\int_{|k|<\lambda k_0}\tfrac{d^Dk}{(2\pi)^D}|\tilde{\phi}(k)|^2 \;= \\ &= \;\tfrac{3}{2}g\tau\tfrac{S_D}{(2\pi)^D}\tfrac{k_0^{(D-4)}(1-\lambda^{(D-4)})}{D-4}\int_{|k|<\lambda k_0}\tfrac{d^Dk}{(2\pi)^D}|\tilde{\phi}(k)|^2 \end{aligned} \tag{7.84}$$

For $D=4-\epsilon$, where $\epsilon \ll 1$, this gives the finite contribution to the parameter $\tilde{\tau}$:

$$\tilde{\tau} \;=\; \tau - \frac{3}{8\pi^2}\tau g\xi \tag{7.85}$$

(we have taken $S_{D=4}=2\pi^2$)

For the diagram (f) of Fig.4 one gets:

$$\begin{aligned} &\tfrac{9}{4}g^2\int_{\lambda k_0<|k|<k_0}\tfrac{d^Dk}{(2\pi)^D}\tfrac{1}{k^4}\int_{|k|<\lambda k_0}\tfrac{d^Dk_1d^Dk_2d^Dk_3d^Dk_4}{(2\pi)^{4D}}\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4) \;= \\ &= \;\tfrac{9}{4}g^2\tfrac{S_D}{(2\pi)^D}\tfrac{k_0^{(D-4)}(1-\lambda^{(D-4)})}{D-4}\int_{|k|<\lambda k_0}\tfrac{d^Dk_1d^Dk_2d^Dk_3d^Dk_4}{(2\pi)^{4D}}\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4) \end{aligned} \tag{7.86}$$

For $D=4-\epsilon$ this gives the following contribution to the parameter $\tilde{g}$:

$$\tilde{g} \;=\; g - \frac{9}{8\pi^2}g^2\xi \tag{7.87}$$

After the operation of rescaling to the original cutoff k_0, according to the eqs.(7.71)-(7.72) for the renormalized parameters $\tau^{(R)}$ and $g^{(R)}$, we get:

$$\begin{aligned} \tau^{(R)} &= (\tau - \tfrac{3}{8\pi^2}\tau g\xi)\exp(2\xi) \\ g^{(R)} &= (g - \tfrac{9}{8\pi^2}g^2\xi)\exp(\epsilon\xi) \end{aligned} \tag{7.88}$$

When $g\xi \ll 1$ and $\epsilon\xi \ll 1$, these equations can be written as follows:

$$\begin{aligned} \ln(\tau^{(R)}) - \ln(\tau) \;&=\; 2\xi - \tfrac{3}{8\pi^2}g\xi \\ g^{(R)} - g \;&=\; \epsilon g\xi - \tfrac{9}{8\pi^2}g^2\xi \end{aligned} \tag{7.89}$$

Assuming that the RG procedure is performed continuously, the evolution (as the scale changes) of the renormalized parameters could be described in terms of the differential equations. From the eqs.(7.89) one obtains:

$$\frac{d\ln|\tau|}{d\xi} \;=\; 2 - \frac{3}{8\pi^2}g \tag{7.90}$$

$$\frac{dg}{d\xi} \;=\; \epsilon g - \frac{9}{8\pi^2}g^2 \tag{7.91}$$

The fixed point solution g^* is defined by the condition $\frac{dg}{d\xi}=0$, which yields:

$$g^* \;=\; \frac{8\pi^2}{9}\epsilon \tag{7.92}$$

Then, from the eq.(7.90) for the scale dimension Δ_τ one finds:

$$\Delta_\tau = 2 - \frac{1}{3}\epsilon \tag{7.93}$$

Correspondingly, according to the eq.(7.60) for the critical exponent ν we obtains:

$$\nu = \frac{1}{2} + \frac{1}{12}\epsilon \tag{7.94}$$

Since the fixed-point value g^* is of the order of ϵ, according to eqs.(7.83), (7.68) and (7.69) there are no corrections in the first order in ϵ to the scale dimensions Δ_ϕ of the field ϕ. Accordingly (see

eqs.(7.50), (7.49)), in the first order in ϵ the critical exponent η, eq.(7.28), of the correlation function $\langle\phi(0)\phi(x)\rangle$ remains zero, as in the Ginzburg-Landau theory.

Using relations (7.29)-(7.33), one can now easily find all the others critical exponents:

$$\begin{array}{lll} \alpha = \frac{1}{6}\epsilon & \gamma = 1 + \frac{1}{6}\epsilon & \beta = \frac{1}{2} - \frac{1}{6}\epsilon \\ \delta \;=\; 3 + \epsilon & \mu \;=\; \frac{1}{3} & \end{array} \tag{7.95}$$

Below we give the values of the critical exponents in the first order in ϵ formally continued for the dimension $D = 3$ ($\epsilon = 1$). These are compared with the corresponding values given by numerical simulations and the Ginzburg-Landau theory:

		ϵ-expansion	Numerical Simulations	Ginzburg-Landau
Order parameter:	β	0.333	0.312 ± 0.003	0.5
	δ	4	5.15 ± 0.02	3
Specific heat:	α	0.167	0.125 ± 0.015	0
Susceptibility:	γ	1.167	1.250 ± 0.003	1
Correlation length	ν	0.583	0.642 ± 0.003	0.5
Correlation function	η	0	0.055 ± 0.010	0

(7.96)

For obtaining results in the second order in ϵ one proceeds in a similar way taking into account next order in ϵ diagrams (see e.g. [25])

It is interesting to note that although the RG ϵ-expansion procedure described above is mathematically not well grounded, it provides rather accurate values for the critical exponents.

7.5 Specific heat singularity in four dimensions

Note also that although in dimensions $D = 4$ the critical exponent α is zero, it does not necessarily mean that the specific heat is not singular at the critical point. Actually in this case the specific heat is logarithmically (and not power-law) divergent. As a matter of useful exercise, let us calculate the specific heat singularity for the four dimensions.

According to the definition of the specific heat (see eqs.(7.34),(7.35)) we have:

$$C = -T\frac{\partial^2 f}{\partial T^2} = \frac{1}{V}\int d^4x \int d^4x' \langle\langle\phi^2(x)\phi^2(x')\rangle\rangle = \int_{|x|<R_c(\tau)} d^4x \langle\langle\phi^2(0)\phi^2(x)\rangle\rangle \tag{7.97}$$

Here the upper cutoff in the spatial integration is taken to be the correlation length, $R_c(\tau) \sim |\tau|^{-1/2}$, since at larger scales all the correlations decay exponentially. The integral in eq.(7.97) can be calculated by summing up the so called "parquette" diagrams [26] shown in Fig.22. The idea of the "parquette" calculations is that all the contributions from the ϕ^4 interactions in the correlation function $\langle\langle\phi^2(x)\phi^2(x')\rangle\rangle$ can be collected into the mass-like vertex $m(\xi)$:

$$\begin{array}{l} C \simeq \int_{|k|>\sqrt{\tau}} \frac{d^4k}{(2\pi)^4} G_0^2(k) (\frac{m(k)}{\tau})^2 \;\sim \int_{|k|>\sqrt{\tau}} \frac{dk}{k} (\frac{m(k)}{\tau})^2 \sim \\ \sim \int_{\xi<\ln(1/\tau)} d\xi (\frac{m(\xi)}{\tau})^2 \end{array} \tag{7.98}$$

Here the renormalization of the "dressed" mass $m(\xi)$ is defined by the diagram shown in Fig.22(b) (see also eqs.(7.84)-(7.87)):

$$m^{(R)} = m - 3mg\int_{\lambda k_0<|k|<k_0} \frac{d^Dk}{(2\pi)^D} G_0^2(k) \;\to m - \frac{3}{8\pi^2} mg\xi \tag{7.99}$$

where, as usual, $\xi \equiv \ln(1/\lambda)$. In differential form:

$$\frac{dm(\xi)}{d\xi} = -\frac{3}{8\pi^2} m(\xi) g(\xi) \tag{7.100}$$

with initial conditions: $m(\xi = 0) = \tau$. The renormalization of the interaction parameter $g(\xi)$ for the dimension $D = 4$ is defined by the RG eq.(7.91) with $\epsilon = 0$:

$$\frac{dg(\xi)}{d\xi} \;=\; -\frac{9}{8\pi^2} g^2(\xi) \tag{7.101}$$

The solution of the eqs.(7.100)-(7.101) is:

$$\begin{aligned} m(\xi) &= \tau(1 + \tfrac{9g}{8\pi^2}\xi)^{-1/3} \\ g(\xi) &= g(1 + \tfrac{9g}{8\pi^2}\xi)^{-1} \end{aligned} \tag{7.102}$$

where $g \equiv g(\xi = 0)$. Then, for the specific heat, eq.(7.98), one gets:

$$\begin{aligned} C(\tau) &\simeq \textstyle\int_{\xi<\ln(1/\tau)} \frac{d\xi}{(1+\frac{9g}{8\pi^2}\xi)^{2/3}} = \\ &= \tfrac{8\pi^2}{3g}[(1 + \tfrac{9g}{8\pi^2}\ln(1/\tau))^{1/3} - 1] \end{aligned} \tag{7.103}$$

This result demonstrates that there exists characteristic temperature interval:

$$\tau_g \sim \exp(-\frac{8\pi^2}{9g}) \;\ll\; 1 \tag{7.104}$$

such that at temperatures not too close to T_c, $\tau_g \ll |\tau| \ll 1$, the system is Gaussian (it does not depend on the non-Gaussian interaction parameter g):

$$C(\tau) \sim \ln(1/\tau) \tag{7.105}$$

This result could be easily obtained just in the framework of the Gaussian Ginzburg-Landau theory:

$$\begin{aligned} &C(\tau) \sim \textstyle\int d^4x \langle\langle \phi^2(0)\phi^2(x)\rangle\rangle \sim \int_{|k|<1} d^4k (k^2 + \tau)^{-2} \sim \\ &\textstyle\int_{\sqrt{\tau}}^1 \frac{d^4k}{k^4} \sim \ln(1/\tau) \end{aligned} \tag{7.106}$$

On the other hand, in the close vicinity of the critical point ($\tau \ll \tau_g$) the theory becomes non-Gaussian, and the result for the specific heat becomes less trivial:

$$C(\tau) \sim \frac{1}{g}(g\ln(1/\tau))^{1/3} \tag{7.107}$$

Thus, although the critical exponent α is zero for the 4-dimensional system, the specific heat still remains (logarithmically) divergent at the critical point.

8 Critical Phenomena in Systems with Disorder

8.1 Harris Criterion

In the studies of the phase transition phenomena the systems considered before were assumed to be perfectly homogeneous. In real physical systems, however, some defects or impurities are always present. Therefore, it is natural to consider what effect impurities might have on the phase transition phenomena. As we have seen in the previous Chapter, the thermodynamics of the second-order phase transition is dominated by large scale fluctuations. The dominant scale, or the correlation length, $R_c \sim |T/T_c - 1|^{-\nu}$ grows as T approach the critical temperature T_c, where it becomes infinite. The large-scale fluctuations lead to singularities in the thermodynamical functions as $|\tau| \equiv |T/T_c - 1| \to 0$. These singularities are the main subject of the theory.

If the concentration of impurities is small, their effect on the critical behavior remains negligible so long as R_c is not too large, i.e. for T not too close to T_c. In this regime the critical behavior will be essentially the same as in the perfect system. However, as $|\tau| \to 0$ ($T \to T_c$) and R_c becomes larger than the average distance between impurities, their influence can become crucial.

As T_c is approached the following change of length scale takes place. First, the correlation length of the fluctuations becomes much larger than the lattice spacing, and the system "forgets" about the lattice. The only relevant scale that remains in the system in this regime is the correlation length $R_c(\tau)$. When we move close to the critical point, R_c grows and becomes larger than the average distance between the impurities, so that the effective concentration of impurities, measured with respect to the correlation length, becomes large. It should be stressed that such a situation is reached for an arbitrary small initial concentration u. The value of u affects only on the width of the temperature region near T_c in which the effective concentration becomes effectively large. If $uR_c^D \gg 1$ there are no grounds for believing that the effect of impurities will be small.

Originally, many years ago, it was generally believed that impurities either completely destroy the long range fluctuations, such that the singularities of the thermodynamical functions are smoothed out [27], [28], or can produce only a shift of a critical point but cannot effect the critical behavior itself, so that the critical exponents remain the same as in the pure system [29]. Later it was realized that an intermediate situation is also possible, in which a new critical behavior, with new critical exponents, is established sufficiently close to the phase transition point [30]. Moreover, a criterion, the so-called Harris criterion, has also been developed, which makes it possible to predict qualitatively the effect of impurities by using the critical exponents of the pure system only [28],[30]. According to this criterion the impurities change the critical behavior only if the specific heat exponent α of the pure system is positive (the specific heat of the pure system is divergent in the critical point). In the opposite case, $\alpha < 0$ (the specific heat if finite), the impurities appear to be irrelevant, i.e. their presence does not affect the critical behavior.

Let us consider this point in more detail. It would be natural to assume that in the ϕ^4-Hamiltonian (Section 7.4) the presence of impurities manifests itself as small random spatial fluctuations of the reduced transition temperature τ. Then near the phase transition point, the D-dimensional Ising-like systems can be described in terms of the scalar field Ginzburg-Landau Hamiltonian with a double-well potential:

$$H = \int d^D x \left[\frac{1}{2} (\nabla \phi(x))^2 + \frac{1}{2} [\tau - \delta\tau(x)] \phi^2(x) + \frac{1}{4} g \phi^4(x) \right] . \tag{8.1}$$

Here the quenched disorder is described by random fluctuations of the effective transition temperature $\delta\tau(x)$ whose probability distribution is taken to be symmetric and Gaussian:

$$P[\delta\tau] = p_0 \exp\left(-\frac{1}{4u} \int d^D x (\delta\tau(x))^2 \right) , \tag{8.2}$$

where $u \ll 1$ is the small parameter which describes the disorder, and p_0 is the normalization constant. For notational simplicity, we define the sign of $\delta\tau(x)$ in eq.(8.1) so that positive fluctuations lead to locally ordered regions, whose effects are the object of our study.

Configurations of the fields $\phi(x)$ which correspond to local minima in H satisfy the saddle-point equation:

$$-\Delta\phi(x) + \tau\phi(x) + g\phi^3(x) = \delta\tau(x)\phi(x) . \tag{8.3}$$

Such localized solutions exist in regions of space where $\tau - \delta\tau(x)$ assumes negative values. Clearly, the solutions of Eq.(8.3) depend on a particular configuration of the function $\delta\tau(x)$ being inhomogeneous. Let us estimate under which conditions the quenched fluctuations of the effective transition temperature are the dominant factor for the local minima field configurations.

Let us consider a large region Ω_L of a linear size $L >> 1$. The spatially average value of the function $\delta\tau(x)$ in this region could be defined as follows:

$$\delta\tau(\Omega_L) = \frac{1}{L^D} \int_{x \in \Omega_L} d^D x \delta\tau(x) . \tag{8.4}$$

Correspondingly, for the characteristic value of the temperature fluctuations (averaged over realizations) in this region we get:

$$\delta\tau_L = \sqrt{\overline{\delta\tau^2(\Omega_L)}} = \sqrt{2u} L^{-D/2} . \tag{8.5}$$

Then, the average value of the order parameter $\phi(\Omega_L)$ in this region can be estimated from the equation:

$$\tau + g\phi^2 = \delta\tau(\Omega_L) \ . \tag{8.6}$$

One can easily see that if the value of τ is sufficiently small, i. e. if

$$\delta\tau(\Omega_L) >> \tau \tag{8.7}$$

then the solutions of Eq.(8.6) are defined only by the value of the random temperature:

$$\phi(\Omega_L) \simeq \pm(\frac{\delta\tau(\Omega_L)}{g})^{1/2} \ . \tag{8.8}$$

Now let us estimate up to which sizes of locally ordered regions this may occur. According to Eq.(8.5) the condition $\delta\tau_L >> \tau$ yields:

$$L << \frac{u^{1/D}}{\tau^{2/D}} \ . \tag{8.9}$$

On the other hand, the estimation of the order parameter in terms of the saddle-point equation (8.6) could be correct only at scales much larger than the correlation length $R_c \sim \tau^{-\nu}$. Thus, one has the lower bound for L:

$$L >> \tau^{-\nu} \ . \tag{8.10}$$

Therefore, quenched temperature fluctuations are relevant when

$$\tau^{-\nu} << \frac{u^{1/D}}{\tau^{2/D}} \tag{8.11}$$

or

$$\tau^{2-\nu D} << u \ . \tag{8.12}$$

According to the scaling relations, eq.(2.55), one has $2 - \nu D = \alpha$. Thus one recovers the Harris criterion: if the specific heat critical exponent of the pure system is positive, then in the temperature interval,

$$\tau < \tau_* \equiv u^{1/\alpha} \tag{8.13}$$

the disorder becomes relevant. This argument identifies $1/\alpha$ as the crossover exponent associated with randomness.

A special consideration is required in the marginal situation $\alpha = 0$. This is the case, for instance, for the four-dimensional ϕ^4-model (Section 7.5), or for the two-dimensional Ising model to be studied in Chapter 5. The calculations show that although the critical exponent of the specific heat remains zero in the impurity models, the logarithmic singularities are effected by the disorder.

8.2 Critical Exponents in the ϕ^4-theory with Impurities

Consider a general case of weakly disordered p-component spin system, which near the critical point, in the continuous limit can be described by the Hamiltonian (cf. eq.(8.1)):

$$H[\delta\tau, \phi] = \int d^D x[\frac{1}{2}\sum_{i=1}^{p}(\nabla\phi_i(x))^2 + \frac{1}{2}(\tau - \delta\tau(x))\sum_{i=1}^{p}\phi_i^2(x) + \frac{1}{4}g\sum_{i,j=1}^{p}\phi_i^2(x)\phi_j^2(x)] \tag{8.14}$$

where the random quantity $\delta\tau(x)$ is described by the Gaussian distribution (8.2).

In terms of the replica approach (Section 1.3) we have to calculate the following replica partition function:

$$\begin{aligned} &Z_n = \overline{(\int D\phi_i(x)\exp\{-H[\delta\tau,\phi]\})^n} = \\ &= \int D\delta\tau(x)\int D\phi_i^a(x)\exp\{-\tfrac{1}{4u}\int d^D x(\delta\tau(x))^2 - \\ &- \int d^D x[\tfrac{1}{2}\textstyle\sum_{i=1}^{p}\sum_{a=1}^{n}(\nabla\phi_i^a(x))^2 + \frac{1}{2}(\tau - \delta\tau(x))\sum_{i=1}^{p}\sum_{a=1}^{n}(\phi_i^a(x))^2 + \\ &+ \tfrac{1}{4}g\textstyle\sum_{i,j=1}^{p}\sum_{a=1}^{n}(\phi_i^a(x))^2(\phi_j^a(x))^2] \end{aligned} \tag{8.15}$$

where the superscript a labels the replicas. (Here and in what follows all irrelevant pre-exponential factors are omitted.) After Gaussian integration over $\delta\tau(x)$ one gets:

$$Z_n = \int D\phi_i^a(x) \exp\{-\int d^D x[\tfrac{1}{2}\sum_{i=1}^p \sum_{a=1}^n (\nabla\phi_i^a(x))^2 + \tfrac{1}{2}\tau \sum_{i=1}^p \sum_{a=1}^n (\phi_i^a(x))^2 + \\ + \tfrac{1}{4}\sum_{i,j=1}^p \sum_{a,b=1}^n g_{ab}(\phi_i^a(x))^2(\phi_j^b(x))^2] \tag{8.16}$$

where

$$g_{ab} = g\delta_{ab} - u \tag{8.17}$$

Now we shell calculate the critical exponents using the RG procedure developed in Section 7.4 for dimension $D = 4-\epsilon$ assuming that $\epsilon \ll 1$. Taking into account the vector and the replica components, the ϕ^4 interaction terms in the Hamiltonian (8.16) could be represented in terms of the diagram shown in Fig.23.

If we proceeding similarly to the calculations of Section 7.4 we find that the (one-loop) renormalization of the interaction parameters g_{ab} (Fig.23) are given by the diagrams shown in Fig.24. Taking into account corresponding combinatoric factors one obtains the following contributions:

$$(a) \rightarrow \; g_{ab}^2 \int_{\lambda k_0<|k|<k_0} \frac{d^D k}{(2\pi)^D} G_0^2(k)|_{\epsilon\ll 1} \simeq g_{ab}^2 \frac{1}{8\pi^2}\ln(\tfrac{1}{\lambda})$$

$$(b) \rightarrow \; \tfrac{1}{2}(g_{aa}+g_{bb})g_{ab} \int_{\lambda k_0<|k|<k_0} \frac{d^D k}{(2\pi)^D} G_0^2(k)|_{\epsilon\ll 1} \simeq \tfrac{1}{2}(g_{aa}+g_{bb})g_{ab} \frac{1}{8\pi^2}\ln(\tfrac{1}{\lambda}) \tag{8.18}$$

$$(c) \rightarrow \; \tfrac{p}{4}\sum_{c=1}^n g_{ac}g_{cb} \int_{\lambda k_0<|k|<k_0} \frac{d^D k}{(2\pi)^D} G_0^2(k)|_{\epsilon\ll 1} \simeq \tfrac{p}{4}\sum_{c=1}^n g_{ac}g_{cb} \frac{1}{8\pi^2}\ln(\tfrac{1}{\lambda})$$

The corresponding RG equations are:

$$\frac{dg_{ab}}{d\xi} = \epsilon g_{ab} - \frac{1}{8\pi^2}[4g_{ab}^2 + 2(g_{aa}+g_{bb})g_{ab} + p\sum_{c=1}^n g_{ac}g_{cb}] \tag{8.19}$$

Taking into account the definition (8.17) one easily gets two RG equations for two interaction parameters $\tilde{g} \equiv g_{aa} = g - u$ and $g_{a\neq b} = -u$:

$$\begin{aligned} \tfrac{d\tilde{g}}{d\xi} &= \epsilon\tilde{g} - \tfrac{1}{8\pi^2}[(8+p)\tilde{g}^2 + p(n-1)u^2] \\ \tfrac{u}{d\xi} &= \epsilon u - \tfrac{1}{8\pi^2}[(4+2p)\tilde{g}u - (4+p(n-2))u^2] \end{aligned} \tag{8.20}$$

In the limit $n \rightarrow 0$ we obtain:

$$\begin{aligned} \tfrac{d\tilde{g}}{d\xi} &= \epsilon\tilde{g} - \tfrac{1}{8\pi^2}[(8+p)\tilde{g}^2 - pu^2] \\ \tfrac{u}{d\xi} &= \epsilon u - \tfrac{1}{8\pi^2}[(4+2p)\tilde{g}u - (4-2p)u^2] \end{aligned} \tag{8.21}$$

Similarly, the renormalization of the "mass" term $\tau(\phi_i^a(x))^2$ is given by the diagrams shown in Fig.25. Their contributions are:

$$\begin{aligned} &(a) \rightarrow \; \tau g_{aa} \int_{\lambda k_0<|k|<k_0} \frac{d^D k}{(2\pi)^D} G_0^2(k)|_{\epsilon\ll 1} \simeq \tau g_{aa} \frac{1}{8\pi^2}\ln(\tfrac{1}{\lambda}) \\ &(b) \rightarrow \; \tfrac{1}{2}p\tau \sum_{c=1}^n g_{ca} \int_{\lambda k_0<|k|<k_0} \frac{d^D k}{(2\pi)^D} G_0^2(k)|_{\epsilon\ll 1} \simeq \tfrac{1}{2}p\tau \sum_{c=1}^n g_{ca} \frac{p}{8\pi^2}\ln(\tfrac{1}{\lambda}) \end{aligned} \tag{8.22}$$

Note that the above contributions does not depend on the replica index a (which for simplicity can be taken to be, for example 1). The corresponding RG equation for the renormalized "mass" τ is:

$$\frac{d\ln|\tau|}{d\xi} = 2 - \frac{1}{8\pi^2}[2g_{aa} + p\sum_{c=1}^n g_{ca}] \tag{8.23}$$

In the limit $n \rightarrow 0$ we finally obtain:

$$\frac{d\ln\tau}{d\xi} = 2 - \frac{1}{8\pi^2}[(2+p)\tilde{g}(\xi) + pu(\xi)] \tag{8.24}$$

where the renormalized interaction parameters $\tilde{g}(\xi)$ and $u(\xi)$ are defined by the eqs.(8.21).

The fixed-point values $\tilde{g}^*$ and u^* are defined by the conditions $\frac{d\tilde{g}^*}{d\xi} = 0$, $\frac{du^*}{d\xi} = 0$, which according to eqs.(8.21) yield:

$$\begin{aligned} (8+p)\tilde{g}^2 - pu^2 &= 8\pi^2\epsilon g \\ (4+2p)\tilde{g}u - (4-2p)u^2 &= 8\pi^2\epsilon u \end{aligned} \tag{8.25}$$

These equations have two non-trivial solutions:

$$\tilde{g}^* = \frac{8\pi^2}{p+8}\epsilon\,; \qquad u^* = 0 \tag{8.26}$$

and

$$\tilde{g}^* = \epsilon\pi^2\frac{p}{2(p-1)}\,; \qquad u^* = \epsilon\pi^2\frac{4-p}{2(p-1)} \tag{8.27}$$

The first solution, eq.(8.26), describes the pure system without disorder. Using eq.(8.23) and the relations (7.60), (7.29) for the critical exponents of the pure system (we mark them by the label "(0)") one gets:

$$\Delta_\tau^{(0)} = 2 - \frac{1}{8\pi^2}(2+p)\tilde{g}^*_{(0)} = 2 - \frac{2+p}{8+p}\epsilon; \;\; \Rightarrow \;\; \nu_{(0)} = \frac{1}{\Delta_\tau^{(0)}} \simeq \frac{1}{2} + \frac{2+p}{4(8+p)}\epsilon \tag{8.28}$$

$$\alpha_{(0)} = 2 - (4-\epsilon)\nu_{(0)} \simeq \frac{4-p}{2(8+p)}\epsilon \tag{8.29}$$

by using relations (7.29)-(7.33) the rest of the exponents are obtained automatically.

Simple analysis of the evolution trajectories defined by the RG eqs.(8.21) near the fixed points (8.26) and (8.27) shows that the "pure" fixed point (8.26) is stable only for $p > 4$. Note that the value of u^* in the other fixed point (8.27) becomes negative for $p > 4$, which means that this fixed-point becomes essentially nonphysical, since the parameter u being a mean square value of the quenched disorder fluctuations is only positively defined.

Thus, the critical behavior of the p-component vector system with $p > 4$ is not modified by the presence of quenched disorder. It should be stressed that it is just the case when the specific heat critical exponent α is negative, eq.(8.29), in accordance with the Harris criteria (Section 8.1).

For $p < 4$ the "pure" fixed point (8.26) becomes unstable and the critical properties of the system is defined by the "random" fixed point given by eq.(8.27). Using eq.(8.23), one gets:

$$\Delta_\tau = 2 - \frac{1}{8\pi^2}[(2+p)\tilde{g}^* + pu^*] = 2 - \frac{3p}{8(p-1)}\epsilon; \;\; \Rightarrow \;\; \nu = \frac{1}{\Delta_\tau} \simeq \frac{1}{2} + \frac{3p}{32(p-1)}\epsilon \tag{8.30}$$

$$\alpha = 2 - (4-\epsilon)\nu \simeq -\frac{4-p}{8(p-1)}\epsilon \tag{8.31}$$

where p must be greater than 1. The rest of the exponents are obtained automatically.

The case of the one-component system, $p = 1$, requires more detailed consideration, because for $p = 1$ the equations (8.21) become degenerate. However, such degeneracy is the property only of the first-order in ϵ approximation. It can be proved that taking into account next-order in ϵ diagrams the degeneracy of the RG equations is removed. It can be shown then that a new "random" fixed-point of the RG equations exists for $p = 1$ as well, and in this case the corrections to the critical exponents appear to be of the order of $\sqrt{\epsilon}$ [30]. We omit this analysis here because it is technically much more cumbersome, while on a qualitative level it provides the results similar to those obtained above.

Thus, in agreement with the Harris criteria (Section 8.1) in the vector p-component system with $p < 4$ the critical behavior is modified by the presence of quenched disorder. In the vicinity of the critical

point a new critical regime appears, and it is described by a new set of (universal) critical exponents. Note that the "random" critical exponent of the specific heat (8.31) appears to be negative, unlike that of the pure system. Therefore, the disorder makes the specific heat to be finite (although still singular) at the critical point, unlike the divergent specific heat of the corresponding pure system.

It should be stressed however, that due to nonperturbative spin-glass phenomena the relevance to real physics of the approach considered in this Section, although it is quite elegant and clear, may be questioned (see next Chapter).

8.3 Critical behavior of the specific heat in four dimensions

In the full analogy with the corresponding considerations for the pure systems (eqs.(7.97) and (7.98), Section 7.5) for the singular part of the specific heat at $D = 4$ we get:

$$C \simeq \int_{|k|>\sqrt{\tau}} \frac{d^4k}{(2\pi)^4} G_0^2(k) (\frac{m(k)}{\tau})^2 \sim \int_{\xi<\ln(1/\tau)} d\xi (\frac{m(\xi)}{\tau})^2 \tag{8.32}$$

Here the renormalization of the "dressed" mass $m(\xi)$ is defined by the "parquette" diagrams of Fig.25. Accordingly, the renormalizations of the interaction parameters $\tilde{g}(\xi)$ and $u(\xi)$ are defined by the RG eqs.(8.21) with $\epsilon = 0$:

$$\frac{d\ln|m|}{d\xi} = -\frac{1}{8\pi^2}[(2+p)\tilde{g} + pu] \tag{8.33}$$

$$\frac{d\tilde{g}}{d\xi} = -\frac{1}{8\pi^2}[(8+p)\tilde{g}^2 - pu^2] \tag{8.34}$$

$$\frac{du}{d\xi} = -\frac{1}{8\pi^2}[(4+2p)\tilde{g}u - (4-2p)u^2] \tag{8.35}$$

The initial conditions are: $m(\xi = 0) = \tau, \tilde{g}(\xi = 0) = g_0, u(\xi = 0) = u_0$.

In the pure system $u = 0$, and the solutions for $m(\xi)$ and $\tilde{g}(\xi) \equiv g(\xi)$ are:

$$\begin{aligned} &g(\xi) = g_0(1 + \tfrac{(8+p)g_0}{8\pi^2}\xi)^{-1}|_{\xi\to\infty} \ \to \sim \tfrac{8\pi^2}{8+p}\xi^{-1} \\ &m(\xi \to \infty) \sim \xi^{-\frac{2+p}{8+p}} \end{aligned} \tag{8.36}$$

Integration in the eq.(8.32) yields the following specific heat singularity:

$$C \sim (\ln(\frac{1}{\tau}))^{\frac{4-p}{8+p}} \tag{8.37}$$

For the system with nonzero disorder interaction parameter u, one finds the following asymptotic (for $\xi \to \infty$) solutions of the eqs.(8.33)-(8.35):

$$\tilde{g}(\xi) \sim \pi^2 \frac{p}{2(p-1)}\xi^{-1}; \ \ u(\xi) \sim \pi^2 \frac{(4-p)}{2(p-1)}\xi^{-1}; \ \ m(\xi) \sim \xi^{-\frac{3p}{8(p-1)}} \tag{8.38}$$

Such solutions exist only for $p < 4$, otherwise u becomes formally negative which is the nonphysical situation. Actually, in this case the vertex $u(\xi)$ is getting zero at a finite scale ξ, and then, the asymptotic solutions for $m(\xi)$ and $\tilde{g}(\xi)$ coincide with the those of the pure system.

The case of one-component field, $p = 1$, requires a special consideration. As the case of the dimension $D = 4 - \epsilon$ (see above) one has to take into account second-order loop terms, which makes the analysis rather cumbersome, and we do not consider it here. On a qualitative level, however, the result for the specific heat appears to be similar to those for $p < 4$: the one-component system with impurities exhibits new type of (logarithmic) singularity.

In the case $p < 4$, the integration in the eq.(8.32) yields:

$$C \sim (\ln(\frac{1}{\tau}))^{-\frac{4-p}{4(p-1)}} \tag{8.39}$$

It is interesting to note that although at the dimension $D = 4$ the critical exponent α of the specific heat is zero, the Harris criterion, taken in the generalized form, still works. Namely, if the specific heat

of the pure system is divergent at the critical point (the case of $p < 4$, eq.(8.37)), the disorder appears to be relevant for the critical behavior, and change the behavior of the specific heat into a new type of (universal) singularity (eq.(8.39)). Otherwise, if the specific heat of the pure system is finite at the critical point ($p > 4$, eq.(8.37)), then the presence of the disorder does not modify the critical behavior.

9 Spin-Glass Effects in Critical Phenomena

9.1 Nonperturbative degrees of freedom

In this Chapter we consider non-trivial spin-glass (SG) effects produced by weak quenched disorder, which have been ignored in the previous Chapter. It will be shown that these effects could dramatically change the whole physical scenario of the critical phenomena.

According to the traditional point of view (considered in the previous Chapter) the effects produced by weak quenched disorder in the critical region could be summarized as follows. If α, the specific heat exponent of the pure system, is greater than zero (i.e. the specific heat of the pure system is divergent at the critical point) the disorder is relevant for the critical behavior, and a new universal critical regime, with new critical exponents, is established sufficiently close to the phase transition point $\tau << \tau_u \equiv u^{1/\alpha}$. In contrast, when $\alpha < 0$ (the specific heat is finite), the disorder appears to be irrelevant, i.e. their presence does not affect the critical behavior. Actually, if the disorder is relevant for the critical behavior, the situation could appear to be much more sophisticated. Let us consider the physical motivation of the traditional RG approach in some more details.

Near the phase transition point the D-dimensional Ising-like systems are described in terms of the scalar field Ginzburg-Landau Hamiltonian with a double-well potential:

$$H = \int d^D x [\frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}(\tau - \delta\tau(x))\phi^2(x) + \frac{1}{4}g\phi^4(x)]. \tag{9.1}$$

Here, as usual, the quenched disorder is described by random fluctuations of the effective transition temperature $\delta\tau(x)$ whose probability distribution is taken to be symmetric and Gaussian:

$$P[\delta\tau] = p_0 \exp\{-\frac{1}{4u}\int d^D x(\delta\tau(x))^2\}, \tag{9.2}$$

where $u \ll 1$ is the small parameter which describes the disorder, and p_0 is the normalization constant.

Now, if one is interested in the critical properties of the system, it is necessary to integrate over all local field configurations up to the scale of the correlation length. This type of calculation is usually performed using a Renormalization Group (RG) scheme, which self-consistently takes into account all the fluctuations of the field on scale lengths up to R_c.

In order to derive the traditional results for the critical properties of this system one can use the usual RG procedure developed for dimensions $D = 4 - \epsilon$, where $\epsilon \ll 1$. Then one finds that in the presence of the quenched disorder the pure system fixed point becomes unstable, and the RG rescaling trajectories arrive to another (universal) fixed point $g_* \neq 0$; $u_* \neq 0$, which yields the new critical exponents describing the critical properties of the system with disorder.

However, there exists an important point which missing in the traditional approach. Consider the ground state properties of the system described by the Hamiltonian (9.1). Configurations of the fields $\phi(x)$ which correspond to local minima in H satisfy the saddle-point equation:

$$-\Delta\phi(x) + (\tau - \delta\tau(x))\phi(x) + g\phi^3(x) = 0\ . \tag{9.3}$$

Clearly, the solutions of this equations depend on a particular configuration of the function $\delta\tau(x)$ being inhomogeneous. The localized solutions with non-zero value of ϕ exist in regions of space where $\tau - \delta\tau(x)$ has negative values. Moreover, one finds a *macroscopic* number of local minimum solutions of the saddle-point equation (9.3). Indeed, for a given realization of the random function $\delta\tau(x)$ there exists a macroscopic number of spatial "islands" where $\tau - \delta\tau(x)$ is negative (so that the local effective temperature is below T_c), and in each of these "islands" one finds two local minimum configurations of the field: one which is "up", and another which is "down". These local minimum energy configurations are separated by finite energy barriers, whose heights increase as the size of the "islands" are increased.

The problem is that the traditional RG approach is only a perturbative theory in which the deviations of the field around the ground state configuration are treated, and it can not take into account other local minimum configurations which are "beyond barriers". This problem does not arise in the pure systems, where the solution of the saddle-point equation is unique. However, in a situation such as that discussed above, when one gets numerous local minimum configurations separated by finite barriers, the direct application of the traditional RG scheme may be questioned.

In a systematic approach one would like to integrate in an RG way over fluctuations around the local minima configurations. Furthermore, one also has to sum over all these local minima up to the scale of the correlation length. In view of the fact that the local minima configurations are defined by the random quenched function $\delta\tau(x)$ in an essentially non-local way, the possibility of implementing such a systematic approach successfully seems rather hopeless.

On the other hand there exists another technique which has been developed specifically for dealing with systems which exhibit numerous local minima states. It is the Parisi Replica Symmetry Breaking (RSB) scheme which has proved to be crucial in the mean-field theory of spin-glasses (see Chapters 3-5). Recent studies show that in certain cases the RSB approach can also be generalized for situations where one has to deal with fluctuations as well [31],[32], [33]. Moreover, recently it has been shown that the RSB technique can be applied successfully for the RG studies of the critical phenomena in the Sine-Gordon model where remarkable instability of the RG flows with respect to the RSB modes has been discovered [34].

It can be argued that the summation over multiple local minimum configurations in the present problem could provide additional non-trivial RSB interaction potentials for the fluctuating fields [35]. Let us consider this point in some more details.

To carry out the appropriate average over quenched disorder one can use the standard replica approach (Sections 1.3 and 8.2). This is accomplished by introducing the replicated partition function, $Z_n \equiv \overline{Z^n[\delta\tau]}$ (see eq.(8.16)):

$$\begin{aligned} Z_n = \int D\phi_a(x) \exp\{-\int d^D x[\tfrac{1}{2}\textstyle\sum_{a=1}^n (\nabla\phi_a(x))^2 + \tfrac{1}{2}\tau \sum_{a=1}^n \phi_a^2(x) \\ + \tfrac{1}{4}\textstyle\sum_{a,b=1}^n g_{ab}\phi_a^2(x)\phi_b^2(x)]\}, \end{aligned} \tag{9.4}$$

where

$$g_{ab} = g\delta_{ab} - u \ . \tag{9.5}$$

is the *replica symmetric* (RS) interaction parameter. If one would start the usual RG procedure for the above replica Hamiltonian (as it is done in Section 8.2), then it would correspond to the perturbation theory around the homogeneous ground state $\phi = 0$.

However, in the situation when there exist numerous local minima solutions of the saddle-point equation (9.3) we have to be more careful. Let us denote the local solutions of the eq.(9.3) by $\psi^{(i)}(x)$ where $i = 1, 2, \dots N_0$ labels the "islands" where $\delta\tau(x) > \tau$. If the size L_0 of an "island" where $(\delta\tau(x) - \tau) > 0$ is not too small, then the value of $\psi^{(i)}(x)$ in this "island" should be $\sim \pm\sqrt{(\delta\tau(x) - \tau)/g}$, where $\delta\tau(x)$ should now be interpreted as the value of $\delta\tau$ averaged over the region of size L_0. Such "islands" occur at a certain finite density per unit volume. Thus the value of N_0 is macroscopic: $N_0 = \kappa V$, where V is the volume of the system and κ is a constant. An approximate global extremal solution $\Phi(x)$ is constructed as the union of all these local solutions, and each local solution can occur with either sign:

$$\Phi_{(\alpha)}[x; \delta\tau(x)] = \sum_{i=1}^{\kappa V} \sigma_i \psi^{(i)}(x) \ , \tag{9.6}$$

where each $\sigma_i = \pm 1$. Accordingly, the total number of global solutions must be $2^{\kappa V}$. We label these solutions by $\alpha = 1, 2, ..., K = 2^{\kappa V}$. As mentioned earlier, it seems unlikely that an integration over fluctuations around $\phi(x) = 0$ will include the contributions from the configurations of $\phi(x)$ which are near a $\Phi(x)$, since $\Phi(x)$ is "beyond a barrier," so to speak. Therefore, it seems appropriate to include separately the contributions from small fluctuations about each of the many $\Phi_{(\alpha)}[x; \delta\tau]$. Thus we have to sum over the K global minimum solutions (non-perturbative degrees of freedom) $\Phi_{(\alpha)}[x; \delta\tau]$ and also to integrate over "smooth" fluctuations $\varphi(x)$ around them

$$
\begin{aligned}
Z[\delta\tau] &= \int D\varphi(x) \sum_{\alpha}^{K} \exp\{-H[\Phi_{(\alpha)} + \varphi; \delta\tau]\} \\
&= \int D\varphi(x) \exp\{-H[\varphi; \delta\tau]\} \times \tilde{Z}[\varphi; \delta\tau] \ ,
\end{aligned}
\tag{9.7}
$$

where

$$
\tilde{Z}[\varphi; \delta\tau] = \sum_{\alpha}^{K} \exp\{-H_{\alpha} - \int d^D x [\frac{3}{2} g \Phi^2_{(\alpha)}(x; \delta\tau)\varphi^2(x) + g\Phi_{(\alpha)}(x; \delta\tau)\varphi^3(x)]\}, \tag{9.8}
$$

and H_{α} is the energy of the α-th solution.

Next we carry out the appropriate average over quenched disorder, and for the replica partition function, Z_n, we get:

$$
Z_n = \int D\delta\tau P[\delta\tau] \int D\varphi_a \exp\{-\sum_{a=1}^{n} H[\varphi_a; \delta\tau]\} \times \tilde{Z}_n[\varphi_a; \delta\tau] \ , \tag{9.9}
$$

where the subscript a is a replica index and

$$
\tilde{Z_n}[\varphi_a; \delta\tau] = \sum_{\alpha_1 \ldots \alpha_n}^{K} \exp\{-\sum_{a}^{n} H_{\alpha_a} - \int d^D x \sum_{a}^{n} [\frac{3}{2} g \Phi^2_{(\alpha_a)}(x; \delta\tau)\varphi^2_a(x) + g\Phi_{(\alpha_a)}(x; \delta\tau)\varphi^3_a(x)]\}. \tag{9.10}
$$

It is clear that if the saddle-point solution is unique, from the eq.(9.9),(9.10) one would obtain the usual RS representation (9.4),(9.5). However, in the case of the macroscopic number of local minimum solutions the problem becomes highly non-trivial.

It is obviously hopeless to try to make a systematic evaluation of the above replicated partition function. The global solutions $\Phi^{(\alpha)}$ are complicated implicit functions of $\delta\tau(x)$. These quantities have fluctuations of two different types. In the first instance, they depend on the stochastic variables $\delta\tau(x)$. But even when the $\delta\tau(x)$ are completely fixed, $\Phi_{(\alpha)}(x)$ will depend on α (which labels the possible ways of constructing the global minimum out of the choices for the signs $\{\sigma\}$ of the local minima). A crude way of treating this situation is to regard the local solutions $\psi^{(i)}(x)$ as if they were random variables, even though $\delta\tau(x)$ has been specified. This randomness, which one can see is not all that different from that which exists in a spin glasses, is the crucial one. It can be shown then, that owing to the interaction of the fluctuating fields with the local minima configurations (the term $\Phi^2_{(\alpha_a)}\varphi^2_a$ in the eq.(9.10)), the summation over solutions in the replica partition function $\tilde{Z_n}[\varphi_a]$, eq.(9.10), could provide the additional non-trivial RSB potential

$$
\sum_{a,b} g_{ab} \varphi^2_a \varphi^2_b
$$

in which the matrix g_{ab} has the Parisi RSB structure [35].

In this Chapter we are going to study the critical properties of weakly disordered systems in terms of the RG approach taking into account the possibility of a general type of the RSB potentials for the fluctuating fields. The idea is that hopefully, as in spin-glasses, this type of generalized RG scheme self-consistently takes into account relevant degrees of freedom coming from the numerous local minima. In particular, the instability of the traditional Replica Symmetric (RS) fixed points with respect to RSB indicates that the multiplicity of the local minima can be relevant for the critical properties in the fluctuation region.

It will be shown (in Section 9.2) that, whenever the disorder appears to be relevant for the critical behavior, the usual RS fixed points (which used to be considered as providing new universal disorder-induced critical exponents) are unstable with respect to "turning on" an RSB potential. Moreover, it will be shown that in the presence of a general type of the RSB potentials, the RG flows actually lead to the so called *strong coupling regime* at the finite spatial scale $R_* \sim \exp(1/u)$ (which corresponds to the temperature scale $\tau_* \sim \exp(-\frac{1}{u})$). At this scale the renormalized matrix g_{ab} develops strong RSB, and the values of the interaction parameters are no longer non-small [36].

Usually the strong coupling situation indicates that certain essentially non-perturbative excitations have to be taken into account, and it could be argued that in the present model these are due to exponentially rare "instantons" in the spatial regions, where the value of $\delta\tau(x) \sim 1$, and the local value of the field $\varphi(x)$ must be $\sim \pm 1$. (A distant analog of this situation exists in the two-dimensional Heisenberg model where the Polyakov renormalization develops into the strong coupling regime at a finite (exponentially large) scale which is known to be due to the nonlinear localized instanton solutions [37]).

In Section 9.3 the physical consequences of the obtained RG solutions will be discussed. In particular we show that due to the absence of fixed points at the disorder dominated scales $R >> u^{-\nu/\alpha}$ (or at the corresponding temperature scales $\tau << u^{1/\alpha}$) there must be no simple scaling of the correlation functions or of other physical quantities. Besides, it is shown that the structure of the SG type two-points correlation functions is characterized by the strong RSB, which indicates on the onset of a new type of the critical behaviour of the SG nature.

The remaining problems as well as future perspectives are discussed in the Section 9.4. Particular attention is given to the possible relevance of the considered RSB phenomena for the so called Griffith phase which is known to exist in a finite temperature interval above T_c [39].

9.2 Replica symmetry breaking in the renormalization group theory

Let us again consider the p-component ferromagnet with quenched random effective temperature fluctuations described by the usual Ginzburg-Landau Hamiltonian, eq.(8.14). In terms of the standard replica approach after integration over the disorder variable $\delta\tau(x)$ for the corresponding replica Hamiltonian we get (see eq.(8.16)):

$$H_n = \int d^D x [\frac{1}{2} \sum_{i=1}^{p} \sum_{a=1}^{n} (\nabla \phi_i^a(x))^2 + \frac{1}{2}\tau \sum_{i=1}^{p} \sum_{a=1}^{n} (\phi_i^a(x))^2 + \frac{1}{4} \sum_{i,j=1}^{p} \sum_{a,b=1}^{n} g_{ab} (\phi_i^a(x))^2 (\phi_j^b(x))^2], \quad (9.11)$$

where $g_{ab} = g\delta_{ab} - u$.

Along the lines of the usual rescaling scheme for the dimension $D = 4 - \epsilon$ (Section 8.2) one gets the following (one-loop) RG equations for the interaction parameters g_{ab} (see eq.(8.19)):

$$\frac{dg_{ab}}{d\xi} = \epsilon g_{ab} - \frac{1}{8\pi^2}(4g_{ab}^2 + 2(g_{aa} + g_{bb})g_{ab} + p\sum_{c=1}^{n} g_{ac}g_{cb}) \ , \quad (9.12)$$

where ξ is the standard rescaling parameter.

Changing $g_{ab} \to 8\pi^2 g_{ab}$, and $g_{a\neq b} \to -g_{a\neq b}$ (so that the off-diagonal elements would be positively defined), and introducing $\tilde{g} \equiv g_{aa}$, we get the following RG equations:

$$\frac{dg_{ab}}{d\xi} = \epsilon g_{ab} - (4 + 2p)\tilde{g}g_{ab} + 4g_{ab}^2 + p \sum_{c\neq a,b}^{n} g_{ac}g_{cb} \qquad (a \neq b), \quad (9.13)$$

$$\frac{d}{d\xi}\tilde{g} = \epsilon\tilde{g} - (8 + p)\tilde{g}^2 - p\sum_{c\neq 1}^{n} g_{1c}^2 \quad (9.14)$$

If one takes the matrix g_{ab} to be replica symmetric, as in the starting form of eq.(9.5), then we can recover the usual RG equations (8.21) for the parameters $\tilde{g}$ and u, and eventually obtain the old results of Section 8.2 for the fixed points and the critical exponents. Here we leave apart the question of how perturbations could arise out of the RS subspace (see also the discussion in [35]) and formally consider the RG eqs.(9.13),(9.14) assuming that the matrix g_{ab} has a general Parisi RSB structure.

According to the standard technique of the Parisi RSB algebra (see Section 3.4), in the limit $n \to 0$ the matrix g_{ab} is parametrized in terms of its diagonal elements $\tilde{g}$ and the off-diagonal *function* $g(x)$ defined in the interval $0 < x < 1$. All the operations with the matrices in this algebra can be performed according to the following simple rules (see eqs.(3.39)-(3.43)):

$$g_{ab}^k \to (\tilde{g}^k; g^k(x)), \quad (9.15)$$

$$(\hat{g}^2)_{ab} \equiv \sum_{c=1}^{n} g_{ac} g_{cb} \to (\tilde{c}; c(x)), \tag{9.16}$$

where

$$\begin{aligned} &\tilde{c} = \tilde{g}^2 - \textstyle\int_0^1 dx g^2(x), \\ &c(x) = 2(\tilde{g} - \textstyle\int_0^1 dy g(y)) g(x) - \textstyle\int_0^x dy [g(x) - g(y)]^2. \end{aligned} \tag{9.17}$$

The RS situation corresponds to the case $g(x) = const$, independent of x.

Using the above rules, from the eqs.(9.13),(9.14) one gets:

$$\frac{d}{d\xi} g(x) = (\epsilon - (4+2p)\tilde{g}) g(x) + 4g^2(x) - 2pg(x) \int_0^1 dy g(y) - p \int_0^x dy (g(x) - g(y))^2 \tag{9.18}$$

$$\frac{d}{d\xi} \tilde{g} = \epsilon \tilde{g} - (8+p)\tilde{g}^2 + p\overline{g^2} \tag{9.19}$$

where $\overline{g^2} \equiv \int_0^1 dx g^2(x)$.

Usually in the studies of the critical behaviour one is looking for the stable fixed-points solutions of the RG equations. The fixed-point values of the of the renormalized interaction parameters are believed to describe the structure of the asymptotic Hamiltonian which allows us to calculate the singular part of the free energy, as well as the other thermodynamic quantities.

From eq.(9.18) one can easily determine what should be the structure of the function $g(x)$ at the fixed point, $\frac{d}{d\xi} g(x) = 0$, $\frac{d}{d\xi} \tilde{g} = 0$. Taking the derivative over x twice, one gets, from Eq.(9.18): $g'(x) = 0$. This means that either the function $g(x)$ is a constant (which is the RS situation), or it has the step-like structure. It is interesting to note that the structure of fixed-point equations is similar to those for the Parisi function $q(x)$ near T_c in the Potts spin-glasses [38], and it is the term $g^2(x)$ in eq.(9.18) which is known to produce 1step RSB solution there. The numerical solution of the RG equations given above demonstrates convincingly that whenever the trial function $g(x)$ has the many-step RSB structure, it quickly develops into the 1-step one with the coordinate of the step being the most right step of the original many-step function.

Let us consider the 1-step RSB ansatz for the function $g(x)$:

$$g(x) = \begin{cases} g_0 & \text{for } 0 \le x < x_0 \\ g_1 & \text{for } x_0 < x \le 1 \end{cases} \tag{9.20}$$

where $0 \le x_0 \le 1$ is the coordinate of the step.

In terms of this ansatz from eqs.(9.18),(9.19) one easily gets the following fixed-point equations for the parameters g_1, g_0 and $\tilde{g}$:

$$\begin{aligned} &(4 - 2px_0) g_0^2 - 2p(1-x_0) g_1 g_0 - (4+2p)\tilde{g} g_0 + \epsilon g_0 = 0 \\ &-px_0 g_0^2 + (4 - 2p + px_0) g_1^2 - (4+2p)\tilde{g} g_1 + \epsilon g_1 = 0 \\ &-px_0 g_0^2 - p(1-x_0) g_1^2 + (8+p)\tilde{g}^2 - \epsilon \tilde{g} = 0. \end{aligned} \tag{9.21}$$

These equations have several non-trivial solutions:

1) The RS fixed-point which corresponds to the pure system, eq.(8.26):

$$g_0 = g_1 = 0; \quad \tilde{g} = \frac{1}{8+p} \epsilon \tag{9.22}$$

This fixed point (in accordance with the Harris criterion) is stable for the number of spin components $p > 4$, and it becomes unstable for $p < 4$.

2) The "random" RS fixed point, eq.(8.27), (for $p > 1$):

$$g_0 = g_1 = \epsilon \frac{4-p}{16(p-1)}; \quad \tilde{g} = \epsilon \frac{p}{16(p-1)}. \tag{9.23}$$

This fixed point was usually considered to be the one which describes the new universal critical behaviour in systems with impurities. This fixed point has been shown to be stable (with respect to the RS deviations!) for $p < 4$, which is consistent with the Harris criterion. (For $p = 1$ this fixed point involves an expansion in powers of $(\epsilon)^{1/2}$ and this structure is only revealed within a two-loop approximation). However, the stability analysis with respect to the RSB deviations shows that this fixed point is *always unstable* [35]. The three eigenvalues of the corresponding linearized equations near this fixed point are:

$$\lambda_1 = -1/2, \quad \lambda_2 = -\frac{(4-p)}{8(p-1)}, \quad \lambda_3 = +\frac{(4-p)}{8(p-1)}$$

so that one of these eigenvalues is always positive. Therefore, whenever the disorder is relevant for the critical behaviour, the RSB perturbations must be the dominant factor in the asymptotic large scale limit.

3) The 1-step RSB fixed point [35]:

$$\begin{aligned} & g_0 = 0; \quad g_1 = \epsilon \tfrac{4-p}{16(p-1)-px_0(8+p)}, \\ & \tilde{g} = \epsilon \tfrac{p(1-x_0)}{16(p-1)-px_0(8+p)}. \end{aligned} \tag{9.24}$$

This fixed point can be shown to be stable (within 1-step RSB subspace!) for:

$$\begin{aligned} & 1 < p < 4, \\ & 0 < x_0 < x_c(p) \equiv \tfrac{16(p-1)}{p(8+p)}. \end{aligned} \tag{9.25}$$

In particular, $x_c(p = 2) = 4/5$; $x_c(p = 3) = 32/33$, and $x_c(p = 4) = 1$. Using the result given by eq.(9.24) one can easily obtain the corresponding critical exponents which become non-universal as they are dependent on the starting parameter x_0 (see Section 9.3). (Note, that in addition to the fixed points listed above there exist several other 1-step RSB solutions which are either unstable or unphysical.)

The problem, however, is that if the parameter x_0 of the starting function $g(x; \xi = 0)$ (or, more generally, the coordinate of the most right step of the many-steps starting function) is taken to be beyond the stability interval, such that $x_c(p) < x_0 < 1$, then there exist *no stable fixed points* of the RG eqs.(9.18),(9.19). One faces the same situation also in the case of a general continuous starting function $g(x; \xi = 0)$. Moreover, according to eq.(9.25) there exist no stable fixed points out of the RS subspace in the most interesting Ising case, $p = 1$.

Unlike the RS situation for $p = 1$, where one finds the stable $\sim \sqrt{\epsilon}$ fixed point in the two-loop RG equations, adding next order terms in the RG equations in the present case does not cure the problem. In the RSB case one finds that in the two-loops RG equations the values of the parameters in the fixed point are formally of the order of one, and this indicates that we are entering the strong coupling regime where all the orders of the RG are getting relevant.

Nevertheless, to get at least some information about the physics behind this instability phenomena, one can proceed to analyse the actual evolution of the above one-loop RG equations. The scale evolution of the parameters of the Hamiltonian would still adequately describe the properties of the system until we reach a critical scale ξ_*, at which the strong coupling regime begins.

The evolution of the renormalized function $g(x; \xi)$ can be analyzed both numerically and analytically. It can be shown (see [36]) that in the case $p < 4$ for a general continuous starting function $g(x; \xi = 0) \equiv g_0(x)$ the renormalized function $g(x; \xi)$ tends to zero everywhere in the interval $0 \leq x < (1 - \Delta(\xi))$, whereas in the narrow (scale dependent) interval $\Delta(\xi)$ near $x = 1$ the values of the function $g(x; \xi)$ increase:

$$g(x; \xi) \sim \begin{cases} a\frac{u}{1-u\xi}; & \text{at } (1-x) << \Delta(\xi) \\ 0; & \text{at } (1-x) >> \Delta(\xi) \end{cases} \tag{9.26}$$

$$\tilde{g}(\xi) \sim u \ln \frac{1}{1-u\xi} \tag{9.27}$$

where

$$\Delta(\xi) \simeq (1-u\xi) \tag{9.28}$$

Here a is a positive non-universal constant, and the critical scale ξ_* is defined by the condition that the values of the renormalized parameters are getting of the order of one: $(1-u\xi_*) \sim u$, or $\xi_* \sim 1/u$. Correspondingly, the spatial scale at which the system enters the strong coupling regime is:

$$R_* \sim \exp(\frac{1}{u}) \tag{9.29}$$

Note that the value of this scale is much greater than the usual crossover scale $\sim u^{-\alpha/\nu}$ (where α and ν are the pure system specific heat and the correlation length critical exponents), at which the disorder is getting relevant for the critical behaviour.

According to the above result, the value of the narrow band near $x = 1$ where the function $g(x;\xi)$ is formally getting divergent is $\Delta(\xi) \simeq (1-u\xi) \to u << 1$ as $\xi \to \xi_*$.

Besides, it can also be shown that the value of the integral

$$\overline{g}(\xi) \equiv \int_0^1 g(x;\xi)$$

becomes formally divergent logarithmically as $\xi \to \xi_*$:

$$\overline{g}(\xi) \sim u \ln \frac{1}{1-u\xi} \tag{9.30}$$

Qualitatively similar asymptotic behaviour for $g(x;\xi)$ is obtained for the case when the starting function $g_0(x)$ has the 1-step RSB structure (9.20), and the coordinate of the step x_0 is in the instability region (or for any x_0 in the Ising case $p = 1$):

$$g(x;\xi) \sim \begin{cases} \frac{g_1(0)}{1-(4-2p+px_0)g_1(0)\xi}; \text{ at } x_0 < x < 1 \\ \\ 0; \quad \text{at } 0 \le x < x_0 \end{cases} \tag{9.31}$$

Here $g_1(0) \equiv g_1(\xi = 0) \sim u$, and the coefficient $(4 - 2p + px_0)$ is always positive. In this case again, the system enters into the strong coupling regime at scales $\xi \sim 1/u$.

Note that the above asymptotics do not explicitly involve ϵ. In fact the role of the parameter $\epsilon > 0$ is to "push" the RG trajectories out of the trivial Gaussian fixed point $g = 0; \tilde{g} = 0$. Thus, the value of ϵ, as well as the values of the starting parameters $g_0(x)$, $\tilde{g}_0$, define a scale at which the solutions finally enter the above asymptotic regime. When $\epsilon < 0$ (above dimensions 4) the Gaussian fixed point is stable; on the other hand, the strong coupling asymptotics still exists in this case as well, separated from the trivial one by a finite (depending on the value of ϵ) barrier. Therefore, although *infinitely small* disorder remains irrelevant for the critical behaviour above the dimension 4, if the disorder is strong enough (bigger than some value depending on the ϵ threshold) the RG trajectories could enter the strong coupling regime again.

9.3 Scaling properties and the replica symmetry breaking

9.3.1 Spatial and temperature scales

The renormalization of the mass term

$$\tau(\xi) \sum_{a=1}^{n} \phi_a^2$$

is described by the following RG equation (see eq.(8.23)):

$$\frac{d}{d\xi} \ln \tau = 2 - \frac{1}{8\pi^2}[(2+p)\tilde{g} + p \sum_{a \neq 1}^{n} g_{1a}] \tag{9.32}$$

Changing (as in the previous Section) $g_{ab} \to 8\pi^2 g_{ab}$, and $g_{a\neq b} \to -g_{a\neq b}$, in the Parisi representation we get:

$$\frac{d}{d\xi}\ln\tau = 2 - [(2+p)\tilde{g}(\xi) + p\int_0^1 g(x;\xi)] \tag{9.33}$$

or

$$\tau(\xi) = \tau_0 \exp\{2\xi - \int_0^\xi d\eta[(2+p)\tilde{g}(\eta) + p\overline{g}(\eta)]\} \tag{9.34}$$

where $\tilde{g}(\eta)$ and $\overline{g}(\eta) \equiv \int_0^1 dx g(x;\eta)$ are the solutions of the RG equations of the previous Section.

Let us first consider the traditional (replica-symmetric) situation. The RS interaction parameters $\tilde{g}(\xi)$ and $g(\xi)$ approach the fixed point values $\tilde{g}_*$ and g_* (which are of the order of ϵ), and then for the dependence of the renormalized mass $\tau(\xi)$, according to (9.34), one gets:

$$\tau(\xi) = \tau_0 \exp\{\Delta_\tau \xi\} \tag{9.35}$$

where

$$\Delta_\tau = 2 - [(2+p)\tilde{g}_* + p g_*] \tag{9.36}$$

At scale ξ_c, such that $\tau(\xi_c)$ is getting of the order of one, the system gets out of the scaling region. Since the RG scale parameter $\xi = \ln R$, where R is the spatial scale, this defines the correlation length R_c as a function of the reduced temperature τ_0. According to (9.35), one obtains:

$$R_c(\tau_0) \sim \tau_0^{-\nu} \tag{9.37}$$

where $\nu = 1/\Delta_\tau$ is the critical exponent of the correlation length.

Actually, if the starting value of the disorder parameter $g(\xi = 0) \equiv u$ is much smaller than starting value of the pure system interaction $\tilde{g}(\xi = 0) \equiv g_0$, the situation is a little bit more complicated. In this case the RG flow for $\tilde{g}(\xi)$ first arrives at the pure system fixed point $\tilde{g}_*^{(pure)}$, as if the disorder perturbation does not exist. Then, since the pure fixed point is unstable with respect to the disorder perturbations, at scales bigger than certain disorder dependent scale ξ_u the RG trajectories eventually arrive at the stable (universal) "random" fixed point ($\tilde{g}_*$, g_*). According to the traditional theory [30] it is known that $\xi_u \sim \frac{\nu}{\alpha}\ln\frac{1}{u}$. The corresponding spatial scale is $R_u \sim u^{-\nu/\alpha}$, and it is big it terms of the small parameter u.

Coming back to the scaling behaviour of the mass parameter $\tau(\xi)$, eq.(9.35), we see that if the value of the temperature τ_0 is such that $\tau(\xi)$ is getting of the order of one before the crossover scale ξ_u is reached, then for the scaling behaviour of the correlation length (as well as for other thermodynamic quantities) one finds essentially the result $R_c(\tau_0) \sim \tau_0^{-\nu_{(pure)}}$ of the pure system. However, critical behaviour of the pure system is observed only until $R_c << R_u$, which imposes the following restriction on the temperature parameter: $\tau_0 >> u^{1/\alpha} \equiv \tau_u$. In other words, at temperatures not too close to T_c, $\tau_u << \tau_0 << 1$, the presence of disorder is irrelevant for the critical behaviour.

On the other hand, if $\tau_0 << \tau_u$ (in the close vicinity of T_c), the RG trajectories for $\tilde{g}(\xi)$ and $g(\xi)$ arrive (after crossover) at a new (universal) "random" fixed point ($\tilde{g}_*$, g_*), and the scaling of the correlation length (as well as other thermodynamic quantities), according to eqs.(9.37)-(9.36), is controlled by a new universal critical exponent ν which is defined by the RS fixed point ($\tilde{g}_*$, g_*) of the random system.

Consider now the situation if the RSB scenario occurred. Again, if the disorder parameter u is small, in the temperature interval $\tau_u << \tau_0 << 1$, the critical behaviour is essentially controlled by the "pure" fixed point, and the presence of disorder is irrelevant. For the same reasons as discussed above, the system gets out of the scaling regime ($\tau(\xi)$ is getting of the order of one) before the disorder parameters start "pushing" the RG trajectories out of the pure system fixed point.

However, at temperatures $\tau_0 << \tau_u$ the situation is completely different from the RS case. If the RG trajectories arrive at the 1-step RSB fixed point, eq.(9.24), (in the $1 < p < 4$ case) then according to the standard scaling relations for the critical exponent of the correlation length one finds:

$$\nu(x_0) = \frac{1}{2} + \frac{1}{2}\epsilon\frac{3p(1-x_0)}{16(p-1) - px_0(p+8)}. \tag{9.38}$$

Thus, depending on the value of the starting parameter x_0 one finds a whole *spectrum* of the critical exponents. Therefore, unlike the traditional point of view described in Section 8.2, the critical properties become *non-universal*, as they are dependent on the concrete statistical properties of the disorder involved. However, this result is not the only consequence of the RSB. More essential effects can be observed in the scaling properties of the spatial correlation functions (see below).

In the Ising case, $p = 1$, as well as in the systems with $1 < p < 4$ for a general starting RSB function $g_0(x)$, the consequences of the RSB appear to be much more dramatic. Here, at scales $\xi >> \xi_u$ (although still $\xi << \xi_* \sim \frac{1}{u}$) according to the solutions (9.26), (9.31) the parameters $\tilde{g}(\xi)$ and $g(x;\xi)$, do not arrive at any fixed point, and they keep evolving as the scale ξ increases. Therefore, in this case, according to eq.(9.34), the correlation length (defined, as usual, by the condition that the renormalized $\tau(\xi)$ is getting of the order of one) is defined by the following non-trivial equation:

$$2\ln R_c - \int_0^{\ln R_c} d\eta[(2+p)\tilde{g}(\eta) + p\overline{g}(\eta)] = \ln\frac{1}{\tau_0} \tag{9.39}$$

Thus, as the temperature becomes sufficiently close to T_c (in the disorder dominated region $\tau_0 << \tau_u$) there will be *no usual scaling dependence* of the correlation length (as well as of other thermodynamic quantities).

Finally, as the temperature parameter τ_0 becomes smaller and smaller, what happens is that at scale $\xi_* \equiv \ln R_* \sim \frac{1}{u}$ we enter the strong coupling regime (such that the parameters $\tilde{g}(\xi)$ and $g(x;\xi)$ are no longer small), while the renormalized mass $\tau(\xi)$ remains still small. The corresponding crossover temperature scale is:

$$\tau_* \sim \exp(-\frac{const}{u}) \tag{9.40}$$

In the close vicinity of T_c at $\tau << \tau_*$ we are facing the situation that at large scales the interaction parameters of the asymptotic (zero-mass) Hamiltonian are no longer small, and the properties of the system cannot be analysed in terms of simple one-loop RG approach. Nevertheless, the qualitative structure of the asymptotic Hamiltonian allows us to argue that in the temperature interval $\tau << \tau_*$ near T_c the properties of the system should be essentially SG-like. The point is that it is the parameter describing the disorder, $g(x;\xi)$, which is the most divergent.

In a sense, here the problem is qualitatively reduced back to the original one with *strong* disorder at the critical point. It doesn't seem probable, however, that the state of the system will be described by non-zero true SG order parameter $Q_{ab} = \langle\phi_a\phi_b\rangle$ (which would mean real SG freezing). Otherwise there must exist finite value of τ at which real thermodynamic phase transition into the SG phase takes place, whereas we observe only the *crossover* temperature τ_*, at which a change of critical regime occurs.

It seems more realistic to expect that at scales $\sim \xi_*$ the RG trajectories finally arrive to a fixed-point characterized by non-small values of the interaction parameters and strong RSB. Then, the SG-like behaviour of the system near T_c will be characterized by highly non-trivial critical properties exhibiting strong RSB phenomena.

9.3.2 Correlation functions

Consider the scaling properties of the spin-glass-type connected correlation function:

$$K(R) = \overline{(\langle\phi(0)\phi(R)\rangle - \langle\phi(0)\rangle\langle\phi(R)\rangle)^2} \equiv \overline{\langle(\phi(0)\phi(R)\rangle)^2} \tag{9.41}$$

In terms of the replica formalism we get:

$$K(R) = \lim_{n\to 0}\frac{1}{n(n-1)}\sum_{a\neq b}^{n} K_{ab}(R) \tag{9.42}$$

where

$$K_{ab}(R) = \langle\langle \phi_a(0)\phi_b(0)\phi_a(R)\phi_b(R)\rangle\rangle \tag{9.43}$$

In terms of the standard RG formalism for the replica correlation function $K_{ab}(R)$ we find that:

$$K_{ab}(R) \sim (G_0(R))^2 (Z_{ab}(R))^2 \tag{9.44}$$

where

$$G_0(R) = R^{-(D-2)} \tag{9.45}$$

is the free-field correlation function, and in the one-loop approximation the scaling of the mass-like object $Z_{ab}(R)$ (with $a \neq b$) is defined by the RG equation:

$$\frac{d}{d\xi} \ln Z_{ab}(\xi) = 2g_{ab}(\xi) \tag{9.46}$$

Here $g_{a\neq b}(\xi) > 0$ is the solution of the corresponding RG equations (9.13)-(9.14), $\xi = \ln R$, and $Z_{ab}(0) \equiv 1$.

For the correlation function (9.44) one finds:

$$K_{ab}(R) \sim (G_0(R))^2 \exp\{4 \int_0^{\ln R} d\xi g_{ab}(\xi)\} \tag{9.47}$$

Correspondingly, in the Parisi representation: $g_{a\neq b}(\xi) \to g(x;\xi)$ and $K_{a\neq b}(R) \to K(x;R)$, one gets:

$$K(x;R) \sim (G_0(R))^2 \exp\{4 \int_0^{\ln R} d\xi g(x;\xi)\} \tag{9.48}$$

To understand the effects of the RSB more clearly let us again consider the situation in the traditional RS case. Here (for $p < 4$) one finds that the interaction parameter $g_{a\neq b}(\xi) \equiv u(\xi)$ arrives at the RS fixed point

$$u_* = \epsilon \frac{4-p}{16(p-1)}$$

and according to eqs.(9.47),(9.42) one obtains the simple scaling:

$$K_{rs}(R) \sim R^{-2(D-2)+\theta} \tag{9.49}$$

with the universal "random" critical exponent

$$\theta = \epsilon \frac{4-p}{4(p-1)} \tag{9.50}$$

In the case of the 1-step RSB fixed point, eq.(9.24), the situation is somewhat more complicated. Here one finds that the correlation function $K(x;R)$ also has 1-step RSB structure:

$$K(x;R) \sim \begin{cases} K_0(R); & \text{for } 0 \leq x < x_0 \\ K_1(R); & \text{for } x_0 < x \leq 1 \end{cases} \tag{9.51}$$

where (in the first order in ϵ)

$$\begin{aligned} K_0(R) &\sim R^{-2(D-2)} = G_0^2(R) \\ K_1(R) &\sim R^{-2(D-2)+\theta_{1rsb}} \end{aligned} \tag{9.52}$$

with the *non-universal* critical exponent θ_{1rsb} explicitly depending on the coordinate of the step x_0:

$$\theta_{1rsb} = \epsilon \frac{4(4-p)}{16(p-1) - px_0(8+p)} \tag{9.53}$$

Since the critical exponent θ_{1rsb} is positive, the leading contribution to the "observable" quantity $K(R) = \overline{\langle\langle \phi(0)\phi(R)\rangle\rangle^2}$, eq.(9.42), is given by $K_1(R)$:

$$K(R) \sim (1-x_0)K_1(R) + x_0 K_0(R) \sim R^{-2(D-2)+\theta_{1rsb}} \tag{9.54}$$

But the difference between the 1-step RSB the RS cases manifests itself not only in the result that their critical exponents θ are different. According to the traditional SG philosophy (Chapter 4), the result that the scaling of the RSB correlation function $K_{ab}(R)$ or $K(x;R)$ does depend on the replica indices (a,b) or the replica parameter x, eq.(9.51), indicates that in different measurements of the correlation function for *the same* realization of the quenched disorder, one is going to obtain *different* results, $K_0(R)$ or $K_1(R)$, with the probabilities defined by the value of x_0.

In real experiments, however, one is dealing with the quantities averaged in space. In particular, for the two-point correlation functions the measurable quantity is obtained by integration over the two points, such that the distance R between them is fixed. Of course, the result obtained this way must be equivalent simply to $K(R)$, eq.(9.54), found by formal averaging over different realizations of disorder, and different scalings $K_0(R)$ and $K_1(R)$ can not be observed this way.

Nevertheless, for somewhat different scheme of the measurements the qualitative difference with the RS situation can be observed. In spin-glasses it is generally believed that RSB can be interpreted as factorization of the phase space into (ultrametric) hierarchy of "valleys", or local minima pure states separated by macroscopic barriers. Although in the present case the local minima configurations responsible for the RSB can not be separated by infinite barriers, it would be natural to interpret the phenomenon observed as effective factorization of the phase space into a hierarchy of valleys separated by *finite* barriers. Since the only relevant scale in the critical region is the correlation length the maximum energy barriers must be proportional to $R_c^D(\tau)$, and they are getting divergent as the critical temperature is approached. In this situation, one could expect that besides the usual critical slowing down (corresponding to the relaxation inside one valley) relaxation times which are qualitatively much bigger would be required for overcoming barriers separated different valleys. Therefore, the traditional measurements of the observables in the "thermal equilibrium" can in fact correspond to the equilibration within one valley only and not to the true thermal equilibrium. Then in different measurements (for the same sample) one could be effectively "trapped" in different valleys and thus the traditional spin-glass situation is recovered.

To check whether the above speculations are correct or not, like in spin-glasses, one can invent traditional "overlap" quantities which could hopefully reveal the existence of the multiple valley structures. For instance, one can introduce the spatially averaged quantity for *pairs* of different realizations of the disorder:

$$K_{ij}(R) \equiv \frac{1}{V}\int d^D r \langle \phi(r)\phi(r+R)\rangle_i \langle \phi(r)\phi(r+R)\rangle_j \tag{9.55}$$

where i and j denote different realizations, and it is assumed that the measurable thermal average corresponds to a particular valley, and not to the true thermal average. If the RS situation occurs (so that only one global valley exists), then for different pairs of realizations one will obtain the same result given by eq.(9.49). On the other hand, in the case of the 1-step RSB, after obtaining statistics over pairs of realizations for $K_{ij}(R)$ one will be getting the result $K_0(R)$ with the probability x_0, and $K_1(R)$ with the probability $(1-x_0)$.

Consider finally what would be the situation if a general type of the RSB takes place. According to the qualitative solution (9.26)-(9.27), the function $g(x;\xi)$ does not arrive at any fixed point at scales $\xi >> \xi_u \sim \frac{\nu}{\alpha}\ln\frac{1}{u}$. Therefore, at the disorder dominated scales $R >> R_u \sim u^{-\nu/\alpha} >> 1$ there must be no scaling behaviour of the correlation function $K(R)$. Near the critical scale $\xi_* \sim 1/u$ the qualitative behaviour of the solution $g(x;\xi)$ is given by eq.(9.26). Therefore, according to eq.(9.48), near the critical scale $R_* \sim \exp(1/u)$ for the correlation function $K(x;R)$ one obtains:

$$K(x;R) \sim \begin{cases} R^{-2(D-2)}(1-u\ln R)^{-4a} \equiv K_1(R); & \text{for } (1-x) << \Delta(R) \\ \\ R^{-2(D-2)} = G_0^2(R) \equiv K_0; & \text{for } (1-x) >> \Delta(R) \end{cases} \tag{9.56}$$

where $\Delta(R) = (1-u\ln R) \to u << 1$ as $R \to R_*$.

At the critical scale we have $(1-u\ln R_*) \sim u$, and according to eq.(9.56) the shape of the replica function $K(x;R)$ must be "quasi-1step":

$$K(x;R_*) \sim \begin{cases} u^{-4a}\exp\{-\frac{2(D-2)}{u}\} \equiv K_1^*; & \text{for } (1-x) << u \\ \\ \exp\{-\frac{2(D-2)}{u}\} \equiv K_0^*; & \text{for } (1-x) >> u \end{cases} \tag{9.57}$$

According to the above discussion of the 1-step RSB case, the result given by eq.(9.57) could be measured for the spatially averaged overlaps of the correlation functions $K_{ij}(R)$, eq.(9.55) in terms of the statistics of the samples realizations. Then, for the correlation function $K_{ij}(R)$ one is expected to obtain the value K_1 with the small probability u and the value K_0 with the probability $(1-u)$. Although both values K_1^* and K_0^* are expected to be exponentially small, their ratio $K_1^*/K_0^* \sim u^{-4a}$ must be large.

Finally, at scales $R >> R_*$ we enter into the strong coupling regime, where simple one-loop RG approach can not no longer be used.

9.3.3 Specific heat

According to the standard procedure the leading singularity of the specific heat can be calculated as follows:

$$C \sim \int d^D R[\overline{\langle\phi^2(0)\phi^2(R)\rangle} - \overline{\langle\phi^2(0)\rangle\langle\phi^2(R)\rangle}] \tag{9.58}$$

In terms of the RG scheme for the correlation function:

$$W(R) \equiv \overline{\langle\phi^2(0)\phi^2(R)\rangle} - \overline{\langle\phi^2(0)\rangle\langle\phi^2(R)\rangle} \tag{9.59}$$

we get:

$$W(R) = (G_0(R))^2 m^2(R) \tag{9.60}$$

where $G_0(R) = R^{-(D-2)}$ is the free field two-point correlation function, and the mass-like object $m(R)$ is given by the solution of the following (one-loop) RG equation (c.f. eq.(9.33)):

$$\frac{d}{d\xi}\ln m(\xi) = -[(2+p)\tilde{g}(\xi) - p\sum_{a\neq 1}^{n} g_{a1}(\xi)] \tag{9.61}$$

Here, as usual, $\xi = \ln R$, and the renormalized interaction parameters $\tilde{g}(\xi)$ and $g_{a\neq b}(\xi)$ are the solutions of the replica RG equations (9.13)-(9.14). In the Parisi representation, $g_{a\neq b}(\xi) \to g(x;\xi)$, one gets:

$$m(R) = \exp\{-(2+p)\int_0^{\ln R} d\xi\tilde{g}(\xi) - p\int_0^{\ln R} d\xi \int_0^1 dx g(x;\xi)\} \tag{9.62}$$

Then, after simple transformations for the singular part of the specific heat, eq.(9.58), we get:

$$C \sim \int_0^{\xi_{max}} d\xi \exp\{\epsilon\xi - 2(2+p)\int_0^{\xi} d\eta\tilde{g}(\eta) - 2p\int_0^{\xi} d\eta\overline{g}(\eta)\} \tag{9.63}$$

where $\overline{g}(\eta) \equiv \int_0^1 dx g(x;\eta)$. The infrared cut-off ξ_{max} in (9.63) is the scale at which the system get out of the scaling regime.

Usually ξ_{max} is the scale at which the renormalized mass $\tau(\xi)$, eq.(9.34), is getting of the order of one, and if the traditional scaling situation takes place, one finds that $\xi_{max} \sim \ln(1/\tau_0)$.

Again, let us first consider the situation in the traditional RS case. Here at scales $\xi >> \xi_u \sim \ln(1/u)$ (which correspond to the temperature region $\tau_0 << \tau_u \sim u^{\nu/\alpha}$) the renormalized parameters $\tilde{g}(\eta)$ and $g(\xi)$ arrive at the universal fixed point $\tilde{g}_* = \epsilon\frac{p}{16(p-1)}$; $g_* = \epsilon\frac{4-p}{16(p-1)}$ given by eq.(9.23), and according to eq.(9.63) for the singular part of the specific heat we find that

$$C(\tau_0) \sim \int_0^{\ln(1/\tau_0)} d\xi \exp\{\xi[\epsilon - 2(2+p)\tilde{g}_* - 2pg_*]\} \sim \tau_0^{\epsilon\frac{4-p}{4(p-1)}} \tag{9.64}$$

So that in the close vicinity of T_c one would expect to observe new universal disorder induced critical behaviour with the negative specific heat critical exponent $\alpha = -\epsilon\frac{4-p}{4(p-1)}$, eq.(3.40) (unlike positive α in the corresponding pure system).

Similarly, if the scenario with the stable 1-step RSB fixed points takes place, then one finds that the specific heat critical exponent $\alpha(x_0)$ becomes non-universal, and depends explicitly on the coordinate of the step x_0 [35]:

$$\alpha(x_0) = -\frac{1}{2}\epsilon\frac{(4-p)(4-px_0)}{16(p-1)-px_0(p+8)}. \tag{9.65}$$

Again, (as for the critical exponent of the correlation length,) depending on the value of the parameter x_0 one finds a whole *spectrum* of the critical exponents. In particular, the possible values of the specific heat critical exponent appear to be in the following band:

$$-\infty < \alpha(x_0) < -\epsilon\frac{(4-p)}{8(p-1)}. \tag{9.66}$$

The upper bound for $\alpha(x_0)$ is achieved in the RS limit $x_0 \to 0$, and it coincides with the usual RS result, eq.(8.31). On the other hand, as x_0 tends to the "border of stability" $x_c(p)$ of the 1-step RSB fixed point, formally the specific heat critical exponent tends to $-\infty$.

In the general RSB case the situation is completely different. Here in the disorder dominated region $\tau_* << \tau_0 << u^{\nu/\alpha}$ (which corresponds to scales $\xi_u << \xi << \xi_*$) the RG trajectories of the interaction parameters $\tilde{g}(\xi)$ and $\overline{g}(\xi)$ do not arrive at any fixed point, and according to eq.(9.64) one finds that the specific heat becomes a complicated function of the temperature parameter τ_0 which does not have the traditional scaling form.

Finally, in the SG-like region in the close vicinity of T_c, where the interaction parameters $\tilde{g}$ and $\overline{g}$ are finite, one finds that the integral over ξ in eq.(9.63) is convergent (so that the upper cutoff scale ξ_{max} becomes irrelevant). Thus, in this case one obtains the result that the "would be singular part" of the specific heat remains finite in the temperature interval $\sim \tau_*$ around T_c, so that the specific heat becomes *non-singular* at the phase transition point.

9.4 Discussion

According to the results obtained in this Chapter, we can conclude that spontaneous replica symmetry breaking coming from the interaction of the fluctuations with the multiple local minima solutions of the mean-field equations has a dramatic effect on the renormalization group flows and on the critical properties. In the systems with the number of spin components $p < 4$ the traditional RG flows at the dimension $D = 4-\epsilon$, which are usually considered as describing the disorder-induced universal critical behavior, appear to be unstable with respect to the RSB potentials as found in spin glasses. For a general type of the Parisi RSB structures there exists no stable fixed points, and the RG flows lead to the *strong coupling regime* at the finite scale $R_* \sim \exp(1/u)$, where u is the small parameter describing the disorder. Unlike the systems with $1 < p < 4$, where there exist stable fixed points having 1-step RSB structures, eq.(9.24), in the Ising case, $p = 1$, there exist no stable fixed points, and any RSB interactions lead to the strong coupling regime.

There exists another general problem which may appear to be interconnected with the RSB phenomena considered in this Chapter. The problem is related to the existence of the so-called Griffith phase [39] in a finite temperature interval above T_c. Numerous experiments for various disordered systems [40] as well as numerical simulations for the three-dimensional random bonds Ising model [41] clearly demonstrate that in the temperature interval $T_c < T < T_0$ (in the high temperature phase) the time correlations decay as $\sim \exp\{-(t/\tau)^\lambda\}$ instead of the usual exponential relaxation law $\sim \exp\{-t/\tau\}$ as it should be in the ordinary paramagnetic phase. Moreover, it is claimed that the parameter λ is the temperature dependent exponent, which is less than unity at $T = T_c$ and which increases monotonically up to $\lambda = 1$ at $T = T_0$. The temperature T_0 is claimed to coincide with the phase transition point of the corresponding pure system.

This phenomenon clearly demonstrates the existence of numerous metastable states separated by finite barriers, their values forming infinite continuous spectrum, and it could be interconnected with

a general idea that the critical phenomena should be described in terms of an infinite hierarchy of correlation lengths and critical exponents [42].

On the other hand, if there is RSB in the fourth-order potential in the problem considered in this Chapter, one could identify a phase with a different symmetry than the conventional paramagnetic phase, and thus there would have to be a temperature T_{RSB} at which this change in symmetry occurs. Actually, the RSB situation is the property of the statistics of the saddle-point solutions only, and it is clear that for large enough τ there must be no RSB. Therefore, one can try to solve the problem of summing over saddle-point solutions for arbitrary τ, aiming to find finite value of τ_c at which the RSB solution for this problem disappears. Of course, in general this problem is very difficult to solve, but one can easily obtain an estimate for the value of τ_c (assuming that at $\tau = 0$ the RSB situation takes place). According to the qualitative study of this problem in the paper [35], the RSB solution can occur only when the effective interactions between the "islands", (where the system is effectively below T_c) is non-small. The islands are the regions where $\delta\tau(r) > \tau$. According to the Gaussian distribution for $\delta\tau(r)$, the average distance between the "islands" must be of the order of $\exp[-\tau^2/u]$, so that the islands become sufficiently remote at $\tau > \sqrt{u}$. The interaction between the islands decreases exponentially with their separation. Therefore at $\tau > \sqrt{u}$ their interaction must be very weak, and there must be no RSB.

Note now that the shift of T_c with respect to the corresponding pure system is also of the order of $\sqrt{u}$. On the other hand, the existence of local solutions to the mean-field equations is reminiscent of the Griffith phase which is claimed to be observed in the temperature interval between T_c of the disordered system and T_c of the corresponding pure system. On these grounds it is tempting to associate the (hypothetical) RSB transition in the statistics of the saddle-point solutions with the Griffith transition. Correspondingly, it would also be natural to suggest that RSB phenomena discovered in the scaling properties of weakly disordered systems could be associated with the Griffith effects.

The other key question which remains unanswered, is whether or not the obtained strong coupling phenomena in the RG flows could be interpreted as the onset of a kind of the spin-glass phase near T_c. Since it is the RSB interaction parameter describing disorder, $g(x;\xi)$, which is the most divergent, it is tempting to argue that in the temperature interval $\tau << \tau_* \sim \exp(-1/u)$ near T_c the properties of the system should be essentially SG-like.

It should be stressed, however, that in the present study we observe only the *crossover* temperature τ_*, at which the change of the critical regime occurs, and it is hardly possible to associate this temperature with any kind of phase transition. Therefore, if the RSB effects could indeed provide any kind of true thermodynamic order parameter, then this must be true in a whole temperature interval where the RSB potentials exist.

The true spin-glass order (in the traditional sense) arises from the onset of the nonzero order parameter $Q_{ab}(x) =< \phi_a(x)\phi_b(x) >; a \neq b$, and, at least for the infinite-range model, Q_{ab} develops the hierarchical dependence on replica indices (Chapter 3). In the present problem we only find that the coupling matrix g_{ab} for the fluctuating fields develops strong RSB structure and its elements become non-small at large scales. Therefore, it seems more realistic to interpret RSB strong coupling phenomena discovered in the RG as a completely new type of the critical behaviour characterized by strong SG-effects in the scaling properties rather then in the ground state.

10 Two-Dimensional Ising Model with Disorder

10.1 Two-dimensional Ising systems

In the general theory of phase transitions the two-dimensional (2D) Ising model plays the prominent role, as it is the simplest nontrivial lattice model with a known exact solution [43]. It is natural to ask, therefore, what effects of quenched disorder is in this particular case. As for the Harris criterion (Section 8.1) the 2D Ising model constitutes a special case, because the specific heat exponent $\alpha = 0$ in this model. However, speaking intuitively, we could expect that like in the case of the vector field model in four dimensions (Section 8.3), the effect of disorder could be predicted on a qualitative level. Although the critical exponent α is zero, the specific heat of the 2D Ising model is (logarithmically) divergent at the critical point. Therefore, we should expect the critical behavior of this system to be strongly effected by the disorder.

Indeed, the exact solution for the critical behavior of the specific heat of the 2D Ising model with a small concentration $c \ll 1$ of impurities [44] (see Section 10.3 below) yields the following result for the singular part of the specific heat:

$$C(\tau) \sim \begin{cases} \ln(\frac{1}{\tau}) & \text{if } \tau^* \ll |\tau| \ll 1 \\ \\ \frac{1}{c}\ln[\ln(\frac{1}{\tau})] & \text{if } \tau \ll \tau^* \end{cases} \tag{10.1}$$

where $\tau^* \sim \exp(-const/c)$ is the temperature scale at which a crossover from one critical behavior to another takes place.

Thus, in the 2D Ising model, as well as in the 4D vector field system, the disorder is relevant. However, unlike the vector field model, the specific heat of the 2D disordered Ising magnet remains divergent at T_c, though the singularity is weakened. Another important property of the 2D Ising model is that unlike the ϕ^4 theory near four dimensions (Chapter 9), the spin-glass RSB phenomena appear to be irrelevant for the critical behavior [52]. Thus, the result given by eq.(10.1) for the leading singularity of the specific heat of the weakly disordered 2D Ising system must be exact.

In this Chapter the emphasis is laid not on the exact lattice expressions, but on their large-scale asymptotics, i.e. we will be interested mainly in the critical long-range behavior because only that is interesting for the general theory of phase transitions. It is well known that in the critical region the 2D Ising model can be reduced to the free-fermion theory [45]. In Section 10.2 this reduction will be demonstrated in very simple terms by means of the Grassman variables technique. The operator language or the transfer matrix formalism will not be used, as they are not symmetric enough to be applied to the model with disorder. The resulting continuum theory, to which the exact lattice disordered model is equivalent in the critical region, appears to be simple enough, and its specific heat critical behavior can be found exactly (Section 10.3).

The results of the recent numerical simulations are briefly described in Section 10.4. General structure of the phase diagram of the disordered 2D Ising model is considered in Section 10.5.

10.2 The fermion solution

The partition function of the pure 2D Ising model is given by:

$$Z \;=\; \sum_{\sigma} \exp\{\beta \sum_{x,\mu} \sigma_x \sigma_{x+\mu}\} \tag{10.2}$$

Here $\{\sigma_x = \pm 1\}$ are the Ising spins defined at lattice sites of a simple square lattice; x are integer valued coordinates of the lattice sites, and $\mu = \mathbf{1}, \mathbf{2}$ are basic vectors of the lattice.

This partition function can be rewritten as follows:

$$\begin{aligned} &Z = \textstyle\sum_\sigma \prod_{x,\mu} \exp\{\beta \sigma_x \sigma_{x+\mu}\} = \sum_\sigma \prod_{x,\mu} (\cosh\beta + \sigma_x \sigma_{x+\mu} \sinh\beta) = \\ &= (\cosh\beta)^V \textstyle\sum_\sigma \prod_{x,\mu} (1 + \lambda \sigma_x \sigma_{x+\mu}) \end{aligned} \tag{10.3}$$

where V is the total number of the lattice bonds, and $\lambda \equiv \tanh\beta$. Expanding the product over the lattice bonds in eq.(10.3) and averaging over the σ's we obtain the following representation for the partition function (the high temperature expansion):

$$Z = (\cosh\beta)^V \sum_{\mathcal{P}} (\lambda)^{L_{\mathcal{P}}} \tag{10.4}$$

The summation here goes over configurations of closed paths $\mathcal{P}$ drawn on lattice links (Fig.26), and $L_{\mathcal{P}}$ is the total length of paths in a particular configuration $\mathcal{P}$.

The summation in the eq.(10.4) could be performed exactly, and these calculations constitute the classical exact solution for the 2D Ising model found by Sherman and Vdovichenko [46]. This solution is well described in detail in textbooks (see e.g. [47]), and we do not consider it here.

Let us now consider an alternative approach to the calculations of the partition function in terms of the so-called Grassmann variables (for detailed treatment of this new mathematics see [48]). The Grassmann variables were first used for the 2D Ising model by Hurst and Green [49], and this approach was later developed by a number of authors [50] (see also [44]). It appears that technically this method enables the equations to be obtained in much simple way. We shall describe this formalism, recover the equation for the partition function, eq.(10.4), and introduce some new notations which will be useful for the problem with disorder.

Let us introduce the four-component Grassmann variables $\{\psi^\alpha(x)\}$ defined at the lattice sites $\{x\}$, where the superscript $\alpha = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ indicates the four directions on the 2D square lattice (such that $\mathbf{3} \equiv -\mathbf{1}$ and $\mathbf{4} \equiv -\mathbf{2}$). All the $\{\psi^\alpha(x)\}$'s and all their differentials $\{d\psi^\alpha(x)\}$ are anticommuting variables; by definition:

$$\begin{aligned} &\psi^\alpha(x)\psi^\beta(y) = -\psi^\beta(y)\psi^\alpha(x) \\ &(\psi^\alpha(x))^2 = 0 \\ &d\psi^\alpha(x)d\psi^\beta(y) = -d\psi^\beta(y)d\psi^\alpha(x) \\ &d\psi^\alpha(x)\psi^\beta(y) = -\psi^\beta(y)d\psi^\alpha(x) \end{aligned} \tag{10.5}$$

and the integration rules are defined as follows:

$$\begin{aligned} &\int d\psi^\alpha(x) = 0 \\ &\int d\psi^\alpha(x)\psi^\alpha(x) = -\int \psi^\alpha(x)d\psi^\alpha(x) = 1 \end{aligned} \tag{10.6}$$

Let us consider the following partition function defined as an integral over all the Grassmann variables of the 2D lattice system:

$$Z = \int D\psi \exp\{A[\psi]\} \tag{10.7}$$

Here the integration measure $D\psi$ and the action $A[\psi]$ are defined as follows:

$$D\psi = \prod_x \left[-d\psi^1(x)d\psi^2(x)d\psi^3(x)d\psi^4(x)\right] \tag{10.8}$$

$$A[\psi] = -\frac{1}{2}\sum_x \overline{\psi}(x)\psi(x) + \frac{1}{2}\lambda \sum_{x,\alpha} \overline{\psi}(x+\alpha)\hat{p_\alpha}\psi(x) \tag{10.9}$$

The "conjugated" variables $\overline{\psi}(x)$ are defined as follows:

$$\overline{\psi^\alpha} = \psi^\gamma(\hat{C}^{-1})^{\gamma\alpha} \tag{10.10}$$

where

$$\hat{C} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}; \quad \hat{C}^{-1} = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \tag{10.11}$$

The vector matrix $\hat{p}_\alpha$ in eq.(10.9) is defined as follows:

$$\hat{p}_\alpha = \{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \} \tag{10.12}$$

More explicitly for the action $A[\psi]$, eq.(10.9), one gets:

$$\begin{aligned} & A[\psi] = -\tfrac{1}{2}\textstyle\sum_x \psi(x)\hat{C}^{-1}\psi(x) + \tfrac{1}{2}\lambda \textstyle\sum_{x,\alpha} \psi(x+\alpha)\hat{C}^{-1}\hat{p}_\alpha\psi(x) \\ & \equiv \textstyle\sum_x \left[\psi^3(x)\psi^1(x) + \psi^4(x)\psi^2(x) + \psi^1(x)\psi^2(x) + \psi^3(x)\psi^4(x) + \psi^2(x)\psi^3(x) + \psi^1(x)\psi^4(x)\right] + \\ & + \lambda \textstyle\sum_x \left[\psi^3(x+1)\psi^1(x) + \psi^4(x+2)\psi^2(x)\right] \end{aligned} \tag{10.13}$$

Using the rules (10.5) and (10.6) one can easily check by direct calculations that the integration in (10.7) with the integration measure (10.8) reproduces the high temperature expansion of the 2D Ising model partition function (10.4) with $\lambda = \tanh\beta$.

Let us consider the Green function:

$$G^{\alpha\beta}(x,x') = Z^{-1}\int D\psi \exp\{A[\psi]\}\psi^\alpha(x)\overline{\psi}^\beta(x') \tag{10.14}$$

Simple (although cumbersome) calculations yield:

$$G^{\alpha\beta}(x,x') = \lambda\sum_\gamma \Lambda^{\alpha\gamma}G^{\gamma\beta}(x-\gamma,x') + \delta_{x,x'}\delta^{\alpha\beta} \tag{10.15}$$

where $\hat{\Lambda} \equiv \sum_\alpha \hat{p}_\alpha$:

$$\hat{\Lambda} = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix} \tag{10.16}$$

If we perform a Fourier transformation of the equation (10.15), it acquires the following matrix form:

$$\hat{G}(k) = (\hat{1} - \lambda\hat{\Lambda}(k))^{-1} \tag{10.17}$$

where

$$\hat{\Lambda}(k) = \sum_\alpha \exp\{-i\mathbf{k}\alpha\}\hat{p}_\alpha = \begin{pmatrix} \exp(-ik_1) & \exp(-ik_2) & 0 & -\exp(ik_2) \\ \exp(-ik_1) & \exp(-ik_2) & \exp(ik_1) & 0 \\ 0 & \exp(-ik_2) & \exp(ik_1) & \exp(ik_2) \\ -\exp(-ik_1) & 0 & \exp(ik_1) & \exp(ik_2) \end{pmatrix} \tag{10.18}$$

It is obvious from eq.(10.17) that, if one of the eigenvalues of the matrix $\lambda\hat{\Lambda}(k)$ becomes unity, it signals a singularity. To find this point we first put the space momentum $k = 0$ (which corresponds to the infinite spatial scale).

The four-valued indices of the Green function $G^{\alpha\beta}$ are related to four possible directions on a square lattice. Therefore, the idea is to perform the Fourier transformation over these angular degrees of freedom. One can easily check that the matrix $\hat{\Lambda}(0)$ diagonalizes in the following representation:

$$\psi_{\pm 1/2} = \frac{1}{2}\begin{pmatrix} 1 \\ \exp(\pm i\frac{\pi}{4}) \\ \exp(\pm i\frac{\pi}{2}) \\ \exp(\pm i\frac{3\pi}{4}) \end{pmatrix},\ \psi_{\pm 3/2} = \frac{1}{2}\begin{pmatrix} 1 \\ \exp(\pm i\frac{3\pi}{4}) \\ \exp(\pm i\frac{3\pi}{2}) \\ \exp(\pm i\frac{9\pi}{4}) \end{pmatrix} \tag{10.19}$$

The transformation matrix from the initial representation to the angular momentum (or spinor) representation with the above basic vectors, has the form:

$$\hat{U} = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{E} & 1 & \frac{1}{E^3} \\ E & \frac{1}{E^2} & E^3 & \frac{1}{E^6} \\ E^2 & \frac{1}{E^3} & E^6 & \frac{1}{E^9} \\ E^3 & \frac{1}{E^4} & E^9 & \frac{1}{E^{12}} \end{pmatrix},\ \ E = \exp(i\frac{\pi}{4}),\ \ \overline{E} = \exp(-i\frac{\pi}{4}) \tag{10.20}$$

In this representation we get:

$$\lambda\hat{\Lambda}'(0) = \lambda\hat{U}^{-1}\hat{\Lambda}(0)\hat{U} = \lambda\begin{pmatrix} \sqrt{2}+1 & 0 & 0 & 0 \\ 0 & \sqrt{2}+1 & 0 & 0 \\ 0 & 0 & -\sqrt{2}+1 & 0 \\ 0 & 0 & 0 & -\sqrt{2}+1 \end{pmatrix} \tag{10.21}$$

There is a singularity in eq.(10.17) (at $k \to 0$) when one of the eigenvalues of $\lambda\hat{\Lambda}'$ becomes unity. From eq.(10.21) we can easily find the critical point of the 2D Ising model:

$$\lambda_c \equiv \tanh\beta_c = \frac{1}{\sqrt{2}+1} \tag{10.22}$$

Another important point which follow from these considerations is that for the critical fluctuations in the vicinity of the critical point only states $\psi_{\pm 1/2}$ (with the eigenvalues $\simeq 1$) are important. Indeed it is easily checked (see below) that the correlation radius for $\psi_{\pm 1/2}$ goes to infinity as $\lambda \to \lambda_c$, while the correlations for $\psi_{\pm 3/2}$ are confined to lattice sizes.

Now, to describe the critical long-range fluctuations, which are responsible for the singularities in the thermodynamical functions, we can expand eq.(10.17) near the point $\lambda = \lambda_c$. Using the explicit expression (10.18), and retaining only the first powers of k and $(\lambda - \lambda_c)/\lambda_c$, one gets:

$$\hat{G}(k) \simeq \frac{2\lambda_c^2}{\Delta}\begin{pmatrix} \tau - ik_1 & \frac{\tau - ik_1 - ik_2}{\sqrt{2}} & -ik_2 & -\frac{\tau - ik_1 + ik_2}{\sqrt{2}} \\ \frac{\tau - ik_1 - ik_2}{\sqrt{2}} & \tau - ik_2 & \frac{\tau + ik_1 - ik_2}{\sqrt{2}} & ik_1 \\ -ik_2 & \frac{\tau + ik_1 - ik_2}{\sqrt{2}} & \tau + ik_1 & \frac{\tau + ik_1 + ik_2}{\sqrt{2}} \\ -\frac{\tau - ik_1 + ik_2}{\sqrt{2}} & ik_1 & \frac{\tau + ik_1 + ik_2}{\sqrt{2}} & \tau + ik_2 \end{pmatrix} \tag{10.23}$$

Here

$$\Delta = det[\hat{1} - \lambda\hat{\Lambda}(k)] \simeq 2\lambda_c^2(\tau^2 + k^2) \tag{10.24}$$

and

$$\tau \equiv 2\frac{(\lambda - \lambda_c)}{\lambda_c} \tag{10.25}$$

In the spinor representation given by eq.(10.19) the asymptotic expression for eq.(10.23) simplifies to the following compact form:

$$\hat{G}_{sp}(k) = \hat{U}^{-1}\hat{G}(k)\hat{U} \simeq \frac{2}{\tau^2 + k^2} \begin{pmatrix} \tau & ik_1 - k_2 & 0 & 0 \\ ik_1 + k_2 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.26}$$

The zero components here are $\sim k^2, \tau^2$. The non-zero 2×2 block can be represented as:

$$\hat{S}(k) = 2\frac{\tau + i\hat{k}}{\tau^2 + k^2} \tag{10.27}$$

Here

$$\hat{k} = k_1\hat{\gamma}_1 + k_2\hat{\gamma}_2; \tag{10.28}$$

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{10.29}$$

The result (10.27) is the Green function of the free (real) spinor field in two Euclidian dimensions described by the Lagrangian:

$$A_{sp}[\psi] = -\frac{1}{4}\int d^2x[\overline{\psi}\hat{\partial}\psi + \tau\overline{\psi}\psi] \tag{10.30}$$

where $\overline{\psi_1} = \psi_2$, and $\overline{\psi_2} = -\psi_1$.

Using eq.(10.30) one immediately finds the logarithmic singularity of the specific heat of the 2D Ising model:

$$\begin{aligned} &Z \simeq \int D\psi \exp\{A_{sp}[\psi]\} \simeq \sqrt{det(\tau + \hat{\partial})}; \\ &F \simeq -\ln Z \simeq -Tr\ln(\tau + \hat{\partial}) \simeq -\int d^2k \ln(\tau^2 + k^2) \sim -\tau^2 \ln\tfrac{1}{|\tau|} \end{aligned} \tag{10.31}$$

Hence

$$C \sim -\frac{d^2}{d\tau^2}F(\tau) \sim \ln\frac{1}{|\tau|} \tag{10.32}$$

10.3 Critical behavior in the disordered model

We turn now to the model with disorder. The partition function of the 2D disoreder Ising model is given by:

$$Z(\beta) = \sum_{\sigma} \exp\left(\beta \sum_{x,\mu} J_{x\mu}\sigma_x\sigma_{x+\mu}\right) \tag{10.33}$$

where the coupling constant $J_{x\mu}$ on a particular lattice bond (x, μ) is equal to the regular value J with probability $(1 - c)$, and to the impurity value $J' \neq J$ with probability c. We impose no restriction on J' but we shall require $c \ll 1$, so that the concentration of impurities is assumed to be small.

The Grassmann variables technique described in the previous Section can be applied to the model with random lattice couplings as well. In this representation the partition function (10.33) is given by:

$$Z(\beta) = \int D\psi \exp\left[-\frac{1}{2}\sum_x \overline{\psi}(x)\psi(x) + \alpha\frac{1}{2}\sum_{x,\mu} \lambda_{x\mu}\overline{\psi}(x + \mu)\hat{p}_\mu\psi(x)\right] \tag{10.34}$$

where

$$\lambda_{x\mu} = \begin{cases} \lambda = \tanh(\beta J) & \text{with probability } 1-c \\ \lambda' = \tanh(\beta J') & \text{with probability } c \end{cases} \tag{10.35}$$

It is easy to check by direct expansion in powers of the second term in (10.34) that the partition function can be represented as a sum over configurations of closed loops, each loop entering with a weight

$$\prod_{\mathcal{P}} \lambda_{x\mu} \Phi(\mathcal{P}) \tag{10.36}$$

where $\Phi(\mathcal{P})$ is an ordered product along the path $\mathcal{P}$ of matrices $\{\hat{p}\}$:

$$\Phi(\mathcal{P}) = \prod_{\mathcal{P}} \hat{p} \tag{10.37}$$

The same representation for the partition function comes from the high temperature expansion of eq.(10.33).

Proceeding along these lines and averaging over the disorder in the couplings one could finally obtain the exact continuum limit representation for the free energy of the impurity model (see [44]). Here, however, we shall consider a more intuitive and much more simplified approach, which, nevertheless, provides the same results as the exact one. This approach is based on the natural assumption that in the continuum limit representation in terms of the free fermion fields (see previous Section) the disorder in the couplings manifests itself as a small spatial disorder in the effective critical temperature τ in the mass term of the spinor Lagrangian (10.30). Therefore, the starting point for further considerations of the disordered model will be the assumption that its continuum limit representation is described by the following spinor Lagrangian:

$$A_{imp}[\psi; \delta\tau(x)] = -\frac{1}{4} \int d^2x [\overline{\psi} \hat{\partial} \psi + (\tau + \delta\tau(x)) \overline{\psi} \psi] \tag{10.38}$$

Here the quenched random variable $\delta\tau(x)$ is assumed to be described by simple Gaussian distribution:

$$P[\delta\tau(x)] = \prod_x [\frac{1}{\sqrt{8\pi u}} \exp\{-\frac{(\delta\tau(x))^2}{8u}\}] \tag{10.39}$$

where the small parameter $u \ll 1$ is proportional to the concentration of impurities.

Then, the self-averaging free energy can be obtained in terms of the traditional replica approach (Section 1.3):

$$F \equiv \overline{F[\delta\tau(x)]} = -\frac{1}{\beta} \lim_{n \to 0} \frac{1}{n} \ln(Z_n) \tag{10.40}$$

where

$$Z_n = \int D\delta\tau(x) \int D\psi^a P[\delta\tau(x)] \exp\left(-\frac{1}{4} \int d^2x \sum_{a=1}^{n} [\overline{\psi^a} \hat{\partial} \psi^a + (\tau + \delta\tau(x)) \overline{\psi^a} \psi^a] \right) \tag{10.41}$$

is the replica partition function and the superscript $a = 1, 2, ..., n$ denotes the replicas. Simple Gaussian integration over $\delta\tau(x)$ yields:

$$Z_n = \int D\psi^a \exp\{A_n[\psi]\} \tag{10.42}$$

where

$$A_n[\psi] = -\int d^2x \left[\frac{1}{4} \sum_{a=1}^{n} \overline{\psi^a} (\hat{\partial} + \tau) \psi^a - \frac{1}{4} u \sum_{a,b=1}^{n} \overline{\psi^a} \psi^a \overline{\psi^b} \psi^b \right] \tag{10.43}$$

Note that rigorous perturbative consideration of the original lattice problem [44] yields the same result for the continuous limit effective Lagrangian (10.43), in which

$$u = c\frac{(\frac{\lambda_c' - \lambda_c}{\lambda_c})^2}{(1 + \frac{1}{2\sqrt{2}}(\lambda_c' - \lambda_c))^2} \tag{10.44}$$

where

$$\begin{aligned} \lambda_c &= \tanh \beta_c J = \sqrt{2} - 1; \\ \lambda_c' &= \tanh \beta_c J' \end{aligned} \tag{10.45}$$

The spinor-field theory with the four-fermion interaction (10.43) obtained above is renormalizable in two dimensions, just as the vector field theory with the interaction ϕ^4 is renormalizable in four dimensions (Sections 7.5 and 8.3).

Indeed, after the scale transformation (see Section 7.3):

$$x \ \rightarrow \ \lambda x \qquad (\lambda > 1) \tag{10.46}$$

one gets:

$$\begin{aligned} &\int d^D x \overline{\psi}(x) \hat{\partial} \psi(x) \ \rightarrow \ \lambda^{D-1} \int d^D x \overline{\psi}(\lambda x) \hat{\partial} \psi(\lambda x) \\ &u \int d^D x (\overline{\psi}(x)\psi(x))(\overline{\psi}(x)\psi(x)) \ \rightarrow \ \lambda^D u \int d^D x (\overline{\psi}(\lambda x)\psi(\lambda x))(\overline{\psi}(\lambda x)\psi(\lambda x)) \end{aligned} \tag{10.47}$$

To leave the gradient term of the Hamiltonian (which is responsible for the scaling of the correlation functions) unchanged, one has to rescale the fields:

$$\psi(\lambda x) \ \rightarrow \ \lambda^{-\Delta_\psi} \psi(x) \tag{10.48}$$

with

$$\Delta_\psi = \frac{D-1}{2} \tag{10.49}$$

The scale dimensions Δ_ψ defines the critical exponent of the correlation function:

$$G(x) = \langle \overline{\psi}(0)\psi(x) \rangle \sim |x|^{-2\Delta_\psi}|_{D=2} = |x|^{-1} \tag{10.50}$$

To leave the Hamiltonian (10.43) unchanged after these transformations one has to rescale the parameter u:

$$u \ \rightarrow \ \lambda^{-\Delta_u} u \tag{10.51}$$

where

$$\Delta_u \ = 2 - D \tag{10.52}$$

Therefore, the scale dimension Δ_u of the four-fermion interaction term is zero in two dimensions, just as the scale dimension of the ϕ^4 interaction term is zero in four dimension.

We shall see below that the renormalization equations lead to the "zero-charge" asymptotics for the charge u and the mass τ. In this lucky case the critical behavior can be found by the renormalization group methods or, in the same way, the main singularities of the thermodynamic functions can be found by summing up the "parquette" diagrams of the theory (10.43) (cf. Section 7.5)

Let us perform the renormalization of the charge u and the mass τ. The diagrammatic representation of the interaction $u(\overline{\psi^a}(x)\psi^a(x))(\overline{\psi^b}(x)\psi^b(x))$ and the mass $\tau(\overline{\psi^a}(x)\psi^a(x))$ terms are shown in Fig.27. It should be stressed that the model under consideration is described in terms of *real* fermions, and although we are using (just for convenience) the notation of the conjugated fields $\overline{\psi}$ they are not

independent variables: $\overline{\psi} = \psi\hat{\gamma}_5$. For that reason the fermion lines in the diagram representation are not be "directed". Actually, the interaction term (Fig.27) can be represented explicitly in terms of only one two-component fermion (unticommuting) field: $u\psi_1^a\psi_2^a\psi_1^b\psi_2^b$. Therefore, the diagonal in replica indeces ($a = b$) interaction terms are identically equal to zero.

Proceeding in a similar way to the calculations of Section 8.2 one then finds that the renormalization of the parameter u is provided only by the diagram shown in Fig.28c, whereas the first two diagrams, Fig.28a and 28b, are identically equal to zero. For the same reason the renormalization of the mass term is provided only by the diagram shown in Fig.29b, while the diagram in Fig.29a is zero. The internal lines in Figs.28 and 29 represent the massless free fermion Green function (cf. eqs.(10.27), (10.28)):

$$\hat{S}_{ab} = -i\frac{\hat{k}}{k^2}\delta_{ab} \tag{10.53}$$

Taking into account corresponding combinatoric factors one easily obtains the following RG transformation for the scale dependent interaction parameter $u(\lambda)$ and mass $\tau(\lambda)$:

$$u^{(R)}(\lambda) = u + 2(n-2)u^2\int_{\lambda k_0<|k|<k_0}\frac{d^2k}{(2\pi)^2}Tr\hat{S}^2(k) \tag{10.54}$$

$$\tau^{(R)}(\lambda) = \tau + 2(n-1)u\tau\int_{\lambda k_0<|k|<k_0}\frac{d^2k}{(2\pi)^2}Tr\hat{S}^2(k) \tag{10.55}$$

Using eq.(10.53) after simple integration one gets the following RG equations (in the limit $n \to 0$):

$$\frac{u(\xi)}{d\xi} = -\frac{2}{\pi}u^2(\xi) \tag{10.56}$$

$$\frac{\ln\tau(\xi)}{d\xi} = -\frac{1}{\pi}u(\xi) \tag{10.57}$$

where, as usual, $\xi \equiv \ln(1/\lambda)$ is the RG parameter. These equations can be easily solved and yield:

$$u(\xi) = \frac{u}{1+\frac{2u}{\pi}\xi} \tag{10.58}$$

$$\tau(\xi) = \frac{\tau}{(1+\frac{2u}{\pi}\xi)^{1/2}} \tag{10.59}$$

where $u \equiv u(\xi = 0)$ and $\tau \equiv \tau(\xi = 0)$. At large scales ($\xi \to \infty$)

$$u(\xi) \sim \frac{1}{\xi} \to 0\,; \quad \tau(\xi) \sim \frac{1}{\sqrt{\xi}} \to 0 \tag{10.60}$$

The critical behavior of a model with the "zero-charge" renormalization can be studied exactly by the RG methods. In a standard way one obtains for the singular part of the specific heat (cf. Section 8.3):

$$C(\tau) \simeq -\frac{1}{2}\int_{|k|>|\tau|}\frac{d^2k}{(2\pi)^2}Tr\hat{S}^2(k)\left(\frac{\tau(k)}{\tau}\right)^2 = \frac{1}{4\pi}\int_{\xi<\ln(1/|\tau|)}\left(\frac{\tau(\xi)}{\tau}\right)^2 \tag{10.61}$$

Here the mass is taken to be dependent on the scale in accordance with eq.(10.59):

$$\left(\frac{\tau(\xi)}{\tau}\right)^2 = \left(1+\frac{2u}{\pi}\xi\right)^{-1} \tag{10.62}$$

Simple calculations yield:

$$C(\tau) \simeq \frac{1}{8u}\ln\left[1+\frac{2u}{\pi}\ln\left(\frac{1}{|\tau|}\right)\right] \tag{10.63}$$

From (10.63) we see that in the temperature range $\tau_u \ll \tau \ll 1$ where

$$\tau_u \sim \exp(-\frac{\pi}{2u}) \tag{10.64}$$

the specific heat has the well known logarithmic behavior of the pure 2D Ising model: $C(\tau) \sim \ln(\frac{1}{|\tau|})$. However, in the close vicinity of the phase transition point, at $|\tau| \ll \tau_u$, the specific heat exhibits different (universal) behavior:

$$C(\tau) \sim \frac{1}{u} \text{lnln} \left(\frac{1}{|\tau|} \right) \tag{10.65}$$

which is still singular, although the singularity is now weaker.

Note that the critical exponent of the two-point correlation function in the 2D Ising model is not modified by the presence of disorder [51]:

$$\overline{\langle \sigma_0 \sigma_x \rangle} \sim |x|^{-1/4} \tag{10.66}$$

This result is also convincingly confirmed by recent numerical simulations [55]-[57].

Note finally, that the effects of the replica symmetry breaking (Chapter 9) in the present case appear to be irrelevant [52]. The corresponding calculations although straightforward, are rather cumbersome and we do not reproduce them here. On the other hand, in the 2D Potts systems the disorder-induced RSB effects can be shown to be relevant and provide the existence of a non-trivial stable fixed point with continuous RSB (for details see [53]).

10.4 Numerical simulations

In recent years extensive numerical investigations on special purpose computers [54] have been performed, with the aim of checking the theoretical results derived for the 2D Ising model with impurity bonds [55],[56],[57]. In these studies, the calculations were performed for the model defined on a square lattice of $L \times L$ spins with the Hamiltonian

$$H = -\sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j \tag{10.67}$$

where the nearest neighbor ferromagnetic couplings J_{ij} are independent random variables taking two values J and J' with probabilities $1-u$ and u correspondingly.

Since the critical behavior of the disordered system is believed to be universal and independent on the concentration of impurities, it is much more convenient in numerical experiments to take the concentration u to be non-small. The point is that according to the theory discussed in the previous section, the parameter u defines the temperature scale $\tau_*(u)$ and correspondingly the spatial scale $L_*(u) \sim \exp\{const/u\}$, eq.(10.64), at which the crossover to the disorder-induced critical behavior takes place. At small concentrations, the crossover scale L_* is exponentially large and it becomes inaccessible in numerical experiments for finite systems. On the other hand, if both coupling constants J and J' are ferromagnetic, then even for a finite concentration of impurity bonds the ferromagnetic ground state (and the ferromagnetic phase transition) is not destroyed, whereas the crossover scale L_* can be expected not to be very large.

Here we shall review only one set of numerical studies in which quite convincing results for the specific heat singularity have been obtained [56]. The model with the concentration of the impurities $u = 1/2$ has been studied. In this particular case the model given by eq.(10.67) appears to be selfdual, and its critical temperature can be determined exactly from the equation [58]:

$$\tanh(\beta_c J) = \exp(-2\beta_c J') \tag{10.68}$$

In the Monte Carlo simulations a cluster-flip algorithm of Swendsen and Wang [59] was used; this algorithm overcomes the difficulty of critical slowing down. In one Monte Carlo sweep, the spin configuration is decomposed into clusters constructed stochastically by connecting neighboring spins of equal sign with the probability $(1 - \exp\{-2\beta J_{ij}\})$. Each cluster is then flipped with probability $1/2$. At T_c and for large lattices, the relaxation to equilibrium for this algorithm appears to be much faster than for the standard single-spin-flip dynamics.

Technically it is much more convenient to calculate the maximum value of the specific heat as the function of the size of the system, instead of the direct dependence of the specific heat from the reduced temperature τ. Since the temperature and the spatial scales are in one to one correspondence

($R_c(\tau) \sim \tau^{-1}$ in the 2D Ising model), the minimum possible value for τ in a finite system of the size L is $\tau_{min} \sim L^{-1}$. Therefore, the maximum value of the specific heat in the system which exhibits the critical behavior $C(\tau)$ must be of the order of $C(L^{-1})$. Then, according to eq.(10.63), the size dependence of the specific heat in the disorder-induced critical regime, in the case of the 2D Ising model, can be expected to be as follows:

$$C(L) = C_0 + C_1 \ln(1 + b\ln(L)) \tag{10.69}$$

where C_0 and C_1 are some constants, and $b = 1/\ln(L_*)$, where L_* is the finite size impurity crossover length.

In general terms, the calculation procedure is as follows. First, one calculates the energy:

$$\overline{\langle H \rangle} = -\frac{1}{L^2}(\overline{\sum_{\langle i,j \rangle} J_{ij}\langle \sigma_i \sigma_j \rangle}) \tag{10.70}$$

where $\langle ... \rangle$ denotes the thermal (Monte Carlo) average. Then the specific heat is obtained from the energy fluctuations:

$$C(L) = L^2 \overline{(\langle H^2 \rangle - \langle H \rangle^2)} \tag{10.71}$$

The simulations were performed for various ratious $r = J'/J = 1/10, 1/4, 1/2$ and 1. The system sizes ranged up to 600×600. Figure 30 displays the data for the critical specific heat, as determined from eq.(10.71) at $r = 1/10, 1/4, 1/2$ and 1, plotted against the logarithm of L. For the sake clarity, the vertical axis has been scaled differently for various r.

For the perfect model, $r = 1$, the deviations from the exactly known asymptotic behavior are obviously rather small for $L \geq 16$, in agreement with the analytic results on the corrections to scaling [60]. At $r = 1/2$ the size dependence data for $L \leq 128$ are still in the perfect Ising regime, where $C \sim \ln(L)$. At $r = 1/4$ and $r = 1/10$ strong deviations from the logarithmic size dependence occur, reflecting the crossover to the randomness-dominated region for sufficiently large values of L.

In Figure 31 the same data are shown plotted against $\mathrm{lnln}(L)$. Strong upwards curvature is evident for $r = 1$ and $1/2$, indicating the logarithmic increase. In notable contrast, the data for $r = 1/4$ approach a straight line for moderate values of L, and those for $r = 1/10$ seems to satisfy such behavior even for small sizes, $L \geq 4$. From fits to eq.(10.69), one obtains $L_* = 16 \pm 4$ at $r = 1/4$ and $L_* = 2 \pm 1$ at $r = 1/10$. The general trends are certainly clear, and confirm the expected crossover to a doubly logarithmic increase of C in the randomness-dominated region sets for smaller sizes L_* as r decreases.

Finally, in Fig.32 the same data for $r = 1/4$ are plotted against $\ln(1 + b\ln(L))$, and exhibit a perfectly straight line for all values of L.

Therefore, in accordance with the analytical predictions of the renormalization group calculations (section 10.3), the results obtained in the Monte-Carlo simulations provide convincing evidences for the onset of a new randomness-dominated critical regime. Besides, evidence is provided for a $\mathrm{lnln}(L)$ dependence in the behavior of the specific heat at the critical point for sufficiently large system sizes.

10.5 General structure of the phase diagram

Let us consider a general structure of the phase diagram of the Ising spin systems with impurities. Apparently, in a ferromagnetic system with antiferromagnetic or broken impurity bonds, as the concentration u of impurities increases, the ferromagnetic phase transition temperature $T_c(u)$ decreases. Then, at some finite concentration u_c the ferromagnetic ground state could be completely destroyed, and correspondingly the phase transition temperature should turn to zero: $T_c(u_c) = 0$. On the basis of these general arguments, one could guess that the qualitative phase diagram of such systems looks like that shown in Fig.33 (for details, see e.g. [61], [62]). To the right of the line $T_c(u)$, the system is either in the paramagnetic state (at high enough temperatures) or in the spin-glass state [63]. The second possibility depends however on the dimensionality of the system; at $D = 2$ the spin-glass state is believed to be unstable at any non-zero temperature [64].

The critical phenomena considered in Section 10.3 formally correspond to the limit of small concentrations of impurities, i.e. they describe the properties of the phase transition near the upper left-hand

side of the line $T_c(u)$ in Fig.33. Nevertheless, the results obtained for the impurity-dominated critical regime appear to be universal, as they are independent of the concentration of impurities (as well as of the values of the impurity bonds). This makes it possible to believe that the critical phenomena in the vicinity of the phase transition line $T_c(u)$ must be the same for other concentrations which are not small. The only parameter which does depend on the impurity concentration is the value of the temperature interval near $T_c(u)$, $\tau_*(u)$, where the impurity dominated critical phenomena occur. According to the analytic theory of section 10.3 the value of this interval shrinks to zero as $u \to 0$: $\tau_*(u) \sim \exp\{-const/u\} \to 0$. At finite concentrations, this temperature interval becomes formally finite, which indicates that the whole critical region near $T_c(u)$ must be described by the impurity-dominated critical regime.

On the other hand it is generally believed [61] that the bottom-right part of the phase transition line $T_c(u)$ (the region near the critical concentration $u = u_c$, $T \ll 1$) belongs to another universality class, which is different from the ferromagnetic phase transition at $u \ll 1$. For example, it is obvious that in magnets with broken impurity bonds the phase transition as a function of the concentration (at $T \ll 1$) at $u = u_c$ must be of the kind of the percolation transition which has nothing to do with the ferromagnetic transition. It means that there must be a special point (T^*, u^*) on the line $T_c(u)$ which separates two different critical regimes.

Actually, there does exist a special line, the so-called Nishimori line $T_N(u)$ [65], which crosses the line $T_c(u)$ at the point (T^*, c^*) (Fig.34). There is no phase transition at the Nishimory line. Formally it is special only in a sense that everywhere on this line the free energy as well as some other thermodynamic quantities appear to be analytic functions of the temperature and the concentration. Moreover, an explicit expression for free energy on the Nishimory line can be obtained for arbitrary T and u at any dimensions. In fact, it makes the structure of the phase diagram much less trivial than that shown in Fig.33. Let us consider this point in more detail.

For the sake of simplicity, let us consider the Ising ferromagnet

$$H = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j \tag{10.72}$$

defined at a lattice with arbitrary structure, where the ferromagnetic spin-spin couplings J_{ij} are equal to 1, while the impurity antiferromagnetic ones are equal to -1, so that the statistical distribution of the J_{ij}'s can be defined as follows:

$$P[J_{ij}] = \prod_{\langle i,j \rangle} \left[(1-u)\delta(J_{ij}-1) + u\delta(J_{ij}+1)\right] \tag{10.73}$$

where u is the concentration of the impurity bonds. One can easily check that the statistical averaging over configurations of the J_{ij}'s:

$$\overline{(...)} = \sum_{J_{ij}=\pm 1} \prod_{\langle i,j \rangle} \left[(1-u)\delta(J_{ij}-1) + u\delta(J_{ij}+1)\right](...) \tag{10.74}$$

can be rewritten as follows:

$$\overline{(...)} = \sum_{J_{ij}=\pm 1} (2\cosh\tilde{\beta}(u))^{-N_b} \exp\left(\tilde{\beta}(u)\sum_{\langle i,j \rangle} J_{ij}\right)(...) \tag{10.75}$$

where N_b is the total number of bonds in the system, and the impurity parameter $\tilde{\beta}(u)$ is defined by the equation:

$$\exp\{-2\tilde{\beta}(u)\} = \frac{u}{1-u} \tag{10.76}$$

For given values of the temperature T and the concentration u the average energy of the system is defined as follows:

$$E(c,T) \equiv \overline{\langle H \rangle} =$$

$$= -(2\cosh\tilde{\beta}(u))^{-N_b} \sum_{J_{ij}=\pm 1} \exp\left(\tilde{\beta}(u) \sum_{\langle i,j\rangle} J_{ij}\right) \frac{\sum_{\sigma=\pm 1}\left(\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j\right) \exp\left(\beta \sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j\right)}{\sum_{\sigma=\pm 1} \exp\left(\beta \sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j\right)} \tag{10.77}$$

It is obvious that the system under consideration is invariant under the local "gauge" transformations:

$$\begin{aligned} \sigma_i &\to \sigma_i s_i \\ J_{ij} &\to J_{ij} s_i s_j \end{aligned} \tag{10.78}$$

for arbitrary $s_i = \pm 1$. Using the above gauge invariance the following trick can be performed. Let us redefine the variables in eq.(10.77) according to (10.78) (which should leave the value of E unchanged), and then let us "average" the obtained expression for E over all configurations of s_i's:

$$E(c,T) = -(2\cosh\tilde{\beta}(u))^{-N_b} 2^{-N} \times$$

$$\times \sum_{J_{ij}=\pm 1} \sum_{s=\pm 1} \exp\left(\tilde{\beta}(u) \sum_{\langle i,j\rangle} J_{ij} s_i s_j\right) \frac{\sum_{\sigma=\pm 1}\left(\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j\right) \exp\left(\beta \sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j\right)}{\sum_{\sigma=\pm 1} \exp\left(\beta \sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j\right)} \tag{10.79}$$

One can easily see that the expression in eq.(10.79)

$$\sum_{s=\pm 1} \exp\left(\tilde{\beta}(u) \sum_{\langle i,j\rangle} J_{ij} s_i s_j\right) \equiv Z[\tilde{\beta}(u), J_{ij}] \tag{10.80}$$

is the partition function of the system at the temperature $\tilde{\beta}(u)$. Therefore, if $\tilde{\beta}(u) = \beta$ the partition function (at the temperature β) in the denominator in the eq.(10.79) is cancelled by the partition function (10.80). In this case the value of the average energy E (as well as the free energy) can be calculated explicitly:

$$E(c,T) = -(2\cosh\tilde{\beta}(u))^{-N_b} 2^{-N} \sum_{J_{ij}=\pm 1} \sum_{\sigma=\pm 1} \left(\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j\right) \exp\left(\beta \sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j\right) =$$

$$= -(2\cosh\tilde{\beta}(u))^{N_b} 2^{-N} \frac{\partial}{\partial\beta} \left[\sum_{J_{ij}=\pm 1} \sum_{\sigma=\pm 1} \exp\left(\beta \sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j\right)\right] =$$

$$= -N_b \tanh\beta(u)$$

$$= -N_b(1 - 2u(T)) \tag{10.81}$$

The internal energy obtained is analytic for all values of the temperature and the concentration.

The above result is valid at the Nishimory line $T_N(u)$ defined by the condition $\tilde{\beta}(u) = \beta$:

$$T_N(u) = \frac{2}{\ln\frac{1-u}{u}} \tag{10.82}$$

This line is shown qualitatively in Fig.34. It starts for the zero concentration (pure system) at $T = 0$, and for $u \to 1/2$ (completely disordered system) $T_N \to \infty$. Apparently, the Nishimory line must cross the phase transition line $T_c(u)$. This creates rather peculiar situation, because at the line of the phase transition the thermodynamic functions should be non-analytic (for details, see [65]). Actually, this crossection point, (T_*, u_*), is argued to be the multicritical point at which the paramagnetic, ferromagnetic and spin-glass phases merge [66].

For the Ising models of this type it can also be proved rigorously [65] that the ferromagnetic phase does not exist for $u > u_*$, where u_* is the point at which the Nishimory line crosses the boundary between the paramagnetic and the ferromagnetic phases $T_c(u)$ (Fig.34). (It means that the structure of the naive phase diagram shown in Fig.33, in general, is not quite correct.) To prove this statement let us consider the following two-point correlation function:

$$G(x) = \overline{\langle \sigma_0 \sigma_x \rangle_\beta} \tag{10.83}$$

where $\langle ... \rangle_\beta$ denotes the thermal average for a given temperature β. Using once again the above trick with the gauge transformation (10.78) for the correlation function (10.83) one gets:

$$G(x) = (2\cosh\tilde{\beta}(u))^{-N_b} \sum_{J_{ij}=\pm 1} \exp\left(\tilde{\beta}(u) \sum_{\langle i,j \rangle} J_{ij}\right) \frac{\sum_{\sigma=\pm 1} (\sigma_0\sigma_x) \exp\left(\beta \sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j\right)}{\sum_{\sigma=\pm 1} \exp\left(\beta \sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j\right)} =$$

$$= (2\cosh\tilde{\beta}(u))^{-N_b} 2^{-N} \times$$

$$\times \sum_{J_{ij}=\pm 1} \sum_{s=\pm 1} (s_0 s_x) \exp\left(\tilde{\beta}(u) \sum_{\langle i,j \rangle} J_{ij} s_i s_j\right) \frac{\sum_{\sigma=\pm 1} (\sigma_0\sigma_x) \exp\left(\beta \sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j\right)}{\sum_{\sigma=\pm 1} \exp\left(\beta \sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j\right)} = \tag{10.84}$$

$$= (2\cosh\tilde{\beta}(u))^{-N_b} 2^{-N} \sum_{s'=\pm 1} \sum_{J_{ij}=\pm 1} \exp\left(\tilde{\beta}(u) \sum_{\langle i,j \rangle} J_{ij} s'_i s'_j\right) \langle (s_0 s_x) \rangle_{\tilde{\beta}(u)} \langle (\sigma_0\sigma_x) \rangle_\beta =$$

$$= \overline{\langle s_0 s_x \rangle_{\tilde{\beta}(u)} \langle \sigma_0 \sigma_x \rangle_\beta}$$

Thus, the absolute value of the correlation function given by eq.(10.83) satisfies the condition:

$$|G(x)| = |\overline{\langle \sigma_0 \sigma_x \rangle_\beta}| \leq \overline{|\langle s_0 s_x \rangle_{\tilde{\beta}(u)}|} \tag{10.85}$$

since the absolute value of any Ising ($|\sigma| = 1$) correlation function does not exceed one. Therefore the absolute value of the two-point correlation function calculated at the temperature T and at the impurity concentration u does not exceed the average of the absolute value of the corresponding correlation function calculated at the Nishimori line at the same impurity concentration. This quantity in the long-range limit $|x| \to \infty$ vanishes if the corresponding point on the Nishimori line is in the paramagnetic phase, which takes place for all concentrations $u > u_*$. On the other hand, the value of the correlation function $G(x)$ in the limit $|x| \to \infty$ becomes the square of the ferromagnetic magnetization: $G(|x| \to \infty) = m^2(T, u)$. Thus, the above simple arguments prove that $m(T, u) \equiv 0$ for $u > u_*$.

Most probably, the boundary line between the ferromagnetic and non-ferromagnetic (spin-glass) phases is vertical to the concentration axis as in Fig.34 [65], although the existence of the reentrant phenomena cannot in general be excluded.

11 The Ising Systems with Quenched Random Fields

11.1 The model

In the previous Chapters we have considered the spin systems in which the quenched disorder was introduced in a form of random fluctuations in the spin-spin interactions. There exists another class of statistical models in which the disorder is present in a form of random magnetic fields. This type of disorder is essentially different from that with fluctuating interactions since external magnetic fields breaks the symmetry with respect to the change of the signs of the spins.

In the most simplified form the random field spin systems could be qualitatively described by the following Ising Hamiltonian:

$$H \;=\; -\sum_{<i\neq j>}^{N} \sigma_i\sigma_j \;-\; \sum_i h_i\sigma_i \tag{11.1}$$

where the Ising spins $\{\sigma_i = \pm 1\}$ are placed in the vertices of a D-dimensional lattice with the ferromagnetic interactions between the nearest neighbors, and the quenched random fields $\{h_i\}$ are described by the symmetric Gaussian distribution:

$$P[h_i] \;=\; \prod_i^N \left[\frac{1}{\sqrt{2\pi h_0^2}} \exp\left(-\frac{h_i^2}{2h_0^2}\right)\right] \;; \quad h_0 << 1 \tag{11.2}$$

The best studied experimentally accessible realizations of systems of this type are the site diluted antiferromagnets in a homogeneous magnetic field [67]. On a qualitative level this could be understood as follows. An ordinary ordered antiferromagnetic system in the ground state is described by the two sublattices, A and B, with magnetizations which are equal in magnitude and opposite in sign. Dilution means that some of the spins chosen at random are removed from both sublattices. In the zero external magnetic field the dilution along does not break symmetry between the two ground states $\sigma_A = -\sigma_B = \pm 1$. However, if the external magnetic field h is nonzero, then an isolated missing spin on the sublattice A provides the energy difference $2h$ between the two ground states $\sigma_A = -\sigma_B = +1$ and $\sigma_A = -\sigma_B = -1$.

Another example is absorbed monolayers with two ground states on impure substrates [68]. Here, if one of the substrate lattice sites is occupied by a quenched impurity it prevents additional occupation of this site, which effectively acts as a local symmetry breaking field. Other realizations are binary liquids in porous media [69], and diluted frustrated antiferromagnets [70].

11.2 General arguments

Despite extensive theoretical and experimental efforts during last twenty years (for reviews see e.g. [71]) there are few reliable statements for the problem of the random field Ising model. According to simple physical arguments by Imry and Ma [72] one would expect that the dimensions above which the ferromagnetic ground state is stable at low temperatures (it is called the lower critical dimension) must be equal to 2. (Note, that for the Ising systems without random fields the low critical dimensions is 1.) Indeed, if we try to reverse a large region Ω of linear size L, there are two competing effects: the gain in energy due to the alignment with the random magnetic field, E_h, and the loss of energy due to the creation of an interface, E_f. The first one scales as follows:

$$E_h \sim \sqrt{\overline{(\sum_{i\in\Omega} h_i)^2}} = \sqrt{\sum_{i,j\in\Omega} \overline{h_i h_j}} \sim h_0 L^{D/2} \tag{11.3}$$

The second one is the energy of a domain wall which is proportional to the square of the boundary of the region Ω:

$$E_f \sim L^{(D-1)} \tag{11.4}$$

These estimates show that at dimensions 2 or lower for arbitrary small (but non-zero) value of the field h_0 the two energies are getting comparable for sufficiently large sizes L, and no spontaneous magnetization should be present. On the other hand, at dimensions greater than 2, the energy at the interface, E_f, is always bigger than E_h. Therefore this effect should not destroy the long range order and a ferromagnetic transition should be present. This naive (but physically correct) argument was later confirmed by a rigorous proof by Imbrie [73].

On the other hand, a perturbative study of the phase transition shows that, as far as the leading large scale divergences are concerned, the strange phenomenon of a dimensional reduction is present, such that the critical exponents of the system in the dimension D are the same as those of the ferromagnetic system without random fields in the dimension d=D-2 [74]. This result would imply that the lower critical dimension is 3, in contradiction with the rigorous results. Actually, the procedure of summing up the leading large scale divergences could give the correct result only if the Hamiltonian in

the presence of the magnetic field has only one minimum. In this case the dimensional reduction can be rigorously shown to be exact, by the use of supersymmetric arguments [75].

However, as soon as the temperature is close enough to the critical point, as well as in a low temperature region, there are values of the magnetic field for which the free energy has more than one minimum (this phenomenon is similar to that considered in Chapter 9). In this situation there is no reason to believe that the supersymmetric approach should give the correct results and therefore the dimensional reduction is not grounded. This is not surprising, because the dimensional reduction completely misses the appearance of the Griffith's singularities [39].

Recently it has also been shown that the existence of more that one solution of the stationary equations in the presence of random fields is related, in the replica approach, to the existence of new instanton-type solutions of the mean-field equations which are not invariant under translations in the replica space [76].

11.3 Griffith phenomena in the low temperature region

In this Section simple physical arguments will be used to demonstrate the origin of the Griffith singularities in the thermodynamical functions in the low-temperature (ordered) phase in the temperature region $h_o^2 << T << 1$ for the dimensions $D < 3$ [77]. This non-perturbative contribution to the thermodynamics will be shown to come from rare, large spin clusters having characteristic size $\sim \sqrt{T}/h_0$ with magnetization opposite to the ferromagnetic background, and which are the *local* minima of the free energy.

If the dimension of the system is greater than 2, then the ground state spin configuration is ferromagnetic. The thermal excitations are the spin clusters with the magnetization opposite to the background. If the linear size L of such cluster is large, then (in the continuous limit) the energy of this thermal excitation could be estimated as follows:

$$E(L) \simeq L^{D-1} - V(L) \tag{11.5}$$

where

$$V(L) = \int_{|x|<L} d^D x \, h(x) \tag{11.6}$$

The statistical distribution of the energy function $V(L)$ (which is the energy of the spin cluster of the size L in the random field $h(x)$) is:

$$P[V(L)] = \int Dh(x) \exp\left(-\frac{1}{2h_0^2}\int d^D x h^2(x)\right) \prod_L \left[\delta\left(\int_{|x|<L} d^D x h(x) - V(L)\right)\right] \tag{11.7}$$

(here and in what follows all kinds of the pre-exponential factors are omitted). For future calculations it will be more convenient to deal with the quenched function $V(L)$ instead of $h(x)$. One can easily derive an explicit expression for the distribution function $P[V(L)]$, eq.(11.7) (for the sake of simplicity the parameter L is first taken to be discrete):

$$P[V(L)] \;=$$

$$= \left[\prod_x \int_{-\infty}^{+\infty} dh(x)\right] \left(\prod_i \int_{-\infty}^{+\infty} d\xi_i\right) \exp\left[-\tfrac{1}{2h_0^2}\int d^D x h^2(x) + i\sum_i \xi_i \left(\int_{|x|<L_i} d^D x h(x) - V(L_i)\right)\right] =$$

$$= \left(\prod_i \int_{-\infty}^{+\infty} d\xi_i\right) \exp\left[-i\sum_i \xi_i V(L_i)\right] \left[\prod_x \int_{-\infty}^{+\infty} dh(x)\right] \times$$

$$\times \exp\left[-\tfrac{1}{2h_0^2}\int d^D x h^2(x) + i\sum_{i=1}^{\infty} \int_{L_i<|x|<L_{i+1}} d^D x h(x) \sum_{j=i}^{\infty} \xi_j\right] \;=$$

$$= \left(\prod_i \int_{-\infty}^{+\infty} d\xi_i\right) \exp\left[-i\sum_i \xi_i V(L_i) - \tfrac{1}{2}h_0^2 \sum_{i=1}^{\infty} (L_{i+1}^D - L_i^D) \left(\sum_{j=i}^{\infty} \xi_j\right)^2\right] =$$

$$\exp\left[-\tfrac{1}{2h_0^2}\sum_i \tfrac{[V(L_{i+1})-V(L_i)]^2}{L_{i+1}^D - L_i^D}\right] \tag{11.8}$$

Making L continuous again, one finally gets:

$$P[V(L)] \;\simeq\; \exp\left[-\frac{1}{2h_0^2}\int dL \frac{1}{L^{D-1}} \left(\frac{dV(L)}{dL}\right)^2\right] \tag{11.9}$$

Since the probability of the flips of big spin clusters is exponentially small, their contributions to the partition function could be assumed to be independent (it is assumed that such clusters are non-interacting, as they are very far from each other). Then, their contribution to the total free energy could be obtained from the statistical averaging of the free energy of one isolated cluster:

$$\Delta F \;=\; -T\left[\prod_L \int dV(L)\right] P[V(L)] \log\left(1 + \int_1^{\infty} dL \exp\{\beta(V(L) - L^{D-1})\}\right) \tag{11.10}$$

Here the factor under the logarithm is the partition function obtained as a sum over all the sizes of the flipped cluster (the factor "1" is the contribution of the ordered state which is the state without the flipped cluster).

The idea of the calculations of the free energy given above is described below. Since at dimensions $D > 2$ the energy $E(L) = L^{D-1} - V(L)$ is on average a function that increases with L, it would be reasonable to expect that the deep local minima (if any) of this function are well separated and the values of the energies at these minima increase with the size L. For this reason, let us assume that the leading contribution in the integration over the sizes of the clusters in eq.(11.10) comes only from *one* (if any) deepest local minimum of the function $L^{D-1} - V(L)$ (for a given realization of the quenched function $V(L)$).

Again, in view of the fact that the energy $E(L) = L^{D-1} - V(L)$ is, on average, the growing function of L, the sufficient condition for existence of a minimum somewhere above a given size L is:

$$\frac{dV(L)}{dL} > (D-1)L^{D-2} \tag{11.11}$$

By the use the above assumptions, the contribution to the free energy from the flipped clusters, eq.(11.10), could be estimated as follows:

$$\Delta F \;\simeq\; -T\int_1^{\infty} dL \int_{-\infty}^{+\infty} dV P_L(V) P\left[\tfrac{dV(L)}{dL} > (D-1)L^{D-2}\right] \times$$
$$\times \log\left[1 + \exp\{\beta(V - L^{D-1})\}\right] \tag{11.12}$$

where $P_L(V)$ is the probability of a given value of the energy V at a given size L, and

$$P\left[\frac{dV(L)}{dL} > (D-1)L^{D-2}\right]$$

is the probability that the condition (11.11) is satisfied at the unit length at the given size L. According to eq.(11.6): $\overline{V^2(L)} \simeq h_0^2 L^D$ (for large values of L). Since the distribution $P_L(V)$ must be Gaussian, one gets:

$$P_L(V) \simeq \exp\{-\frac{V^2}{2h_0^2 L^D}\} \tag{11.13}$$

Note that the above result can also be obtained by integrating the general distribution function $P[V(L)]$, eq.(11.9), over all the "trajectories" $V(L)$ with the fixed value $V(L) = V$ at the given length L.

The value of the probability $P\left[\frac{dV(L)}{dL} > (D-1)L^{D-2}\right]$ could also be obtained by integrating $P[V(L)]$ over all the functions $V(L)$ conditioned by $\frac{dV(L)}{dL} > (D-1)L^{D-2}$ (at the given value of L). It is clear, however, that with the exponential accuracy, the result of such an integration is defined only by the lower bound $(D-1)L^{D-2}$ for the derivative $dV(L)/dL$ (at the given length L) in eq.(11.9). Therefore, one gets:

$$P\left[\frac{dV(L)}{dL} > (D-1)L^{D-2}\right] \simeq \exp\left[-\frac{\left((D-1)L^{D-2}\right)^2}{2h_0^2 L^{D-1}}\right] = \exp\left[-\frac{(D-1)^2 L^{D-3}}{2h_0^2}\right] \tag{11.14}$$

Note the important property of the energy $E(L)$, which follows from the eqs.(11.13)-(11.14): although at dimensions $D > 2$ the function $E(L)$ increases with L, the probability of finding a local minimum of this function at dimensions $D < 3$ also increases with L. It is the competition of these two effects which produces the non-trivial contribution to be calculated below.

In the limit of low temperatures, $T << 1$ (although still $T >> h_0^2$), the contribution to the free energy, eq.(11.12), could be divided into two separate parts:

$$\begin{aligned}
&\Delta F = \Delta F_1 \ + \ \Delta F_2 \ \simeq \\
&-T\int_1^\infty dL \int_{V>L^{D-1}} dV \exp\left[-\tfrac{V^2}{2h_0^2 L^D} - \tfrac{(D-1)^2 L^{D-3}}{2h_0^2}\right] \log(1+\exp\{\beta(V-L^{D-1})\}) - \\
&-T\int_1^\infty dL \int_{V<L^{D-1}} dV \exp\left[-\tfrac{V^2}{2h_0^2 L^D} - \tfrac{(D-1)^2 L^{D-3}}{2h_0^2}\right] \log(1+\exp\{\beta(V-L^{D-1})\})
\end{aligned} \tag{11.15}$$

The first one is the contribution from the minima which have negative energies (the excitations which produce the *gain* in energy with respect to the ordered state). Here the leading contribution in the integration over V comes from the limit $V = L^{D-1}$, and in the leading order one gets:

$$\Delta F_1 \ \sim \ -T\int_1^\infty dL \exp\left[-\frac{L^{D-2}}{2h_0^2} - \frac{(D-1)^2 L^{D-3}}{2h_0^2}\right] \tag{11.16}$$

For dimensions $D > 2$ the leading contribution to ΔF_1 comes from $L \sim 1$ and this take us back to the Imry and Ma [72] arguments that there are no flipped big spin clusters which would produce the gain in energy with respect to the ordered state.

The second contribution in eq.(11.15) comes from the local minima which have positive energies. These could contribute to the free energy only as a thermal excitations at non-zero temperatures. In the limit of low temperatures $\beta >> 1$ one can approximate:

$$\log\left[1+\exp\{\beta(V-L^{D-1})\}\right] \ \simeq \ \exp\left[-\beta(L^{D-1}-V)\right] \tag{11.17}$$

where $L^{D-1} > V$. Then, for ΔF_2 one gets:

$$\Delta F_2 \simeq -T \int_1^\infty dL \int_{-\infty}^{L^{D-1}} dV \exp\left[-\frac{V^2}{2h_0^2 L^D} - \frac{(D-1)^2 L^{D-3}}{2h_0^2} + \beta V - \beta L^{D-1}\right] \tag{11.18}$$

The main contribution in this integral also comes from the "trivial" region $L \sim 1$ and $V \sim \beta h_0^2$, which corresponds to the "elementary excitations" at scales of the lattice spacing. However, if the temperature is not too low: $\beta h_0^2 << 1$ and $D < 3$, there exists another non-trivial contribution which comes from the vicinity of the saddle point:

$$\begin{aligned} V_* &= (\beta h_0^2) L_*^D \\ L_* &= \sqrt{\frac{(D-1)(3-D)}{2\beta h_0^2}} >> 1 \end{aligned} \tag{11.19}$$

which is separated from the region $L \sim 1, V \sim \beta h_0^2$ by a large barrier. Note that the condition of integration in eq.(11.18), $V_* << L_*^{D-1}$, according to eq.(11.19) is satisfied for $L_* << 1/\beta h_0^2$, which is correct only if $\beta h_0^2 << 1$.

For the contribution to the free energy at this saddle-point one gets:

$$\Delta F_2 \sim \exp\left[-\frac{const}{2h_0^2}(\beta h_0^2)^{\frac{3-D}{2}}\right] \tag{11.20}$$

where

$$const = \frac{1}{2}(D+1)(D-1)^{\frac{D-1}{2}}\left(\frac{2}{3-D}\right)^{\frac{3-D}{2}} \tag{11.21}$$

The result (11.20) demonstrates that in addition to the usual thermal excitations in the vicinity of the ordered state (which could be taken into account by the traditional perturbation theory), due to the interaction with the random fields there exist essentially non-perturbative large-scale thermal excitations which produce exponentially small non-analytic contribution to the thermodynamics. These excitations are large spin clusters with the magnetization opposite to the background which are the *local* energy minima. At finite temperatures such that $h_o^2 << T << 1$ the characteristic size of the clusters giving the leading contribution to the free energy is $L_* \sim \sqrt{T}/h_0 >> 1$.

This phenomenon, although seems to produce negligibly small contribution to the thermodynamical functions, could be extremely important for understanding the dynamical relaxation processes. The large clusters with reversed magnetization being the local minima, are separated from the ground state by large energy barriers, and this could produce the essential slowing down of the relaxation (see e.g. [78]). In particular, the characteristic "saddle-point" clusters (eq.(11.19)) with the size $L_*(T) \sim \sqrt{T}/h_0 >> 1$ are separated from the ground state by the energy barrier of the order of $V_* \sim (\beta h_0^2)^{-(D-2)/2} >> 1$, and the corresponding characteristic relaxation time at low temperatures can be expected to be exponentially large:

$$\tau(T) \sim \exp\left[\beta(\beta h_0^2)^{-\frac{D-2}{2}}\right] >> 1 \tag{11.22}$$

However, in order to describe the time asymptotics of the relaxation processes one needs to know the *spectrum* of the relaxation times (or the energy barriers), and this would require more special consideration.

Unfortunately, the results obtained in this Section can not be applied directly for the dimension $D = 3$, which appears to be marginal for the considered phenomena (at dimensions $D > 3$ this type of the non-perturbative effects are absent). At $D = 3$ all those simple estimates for the energies and probabilities of the cluster excitations which have been used in this Section (in particular, eq.(11.14)) do not work, and much more detailed analysis is required.

On the other hand, it seems quite reasonable to expect that the results obtained are correct at dimensions $D = 2$ regardless of the fact that the long-range order in not stable there. The point is that at $D = 2$ the correlation length at which the long-range order is destroyed is exponentially large in the parameter $1/h_0$, whereas the characteristic size of the spin clusters considered here is only the power of the parameter $1/h_0$. Therefore, at the scales at which the Griffith singularities (eq.(11.20)) appear, the system is still effectively ordered at $D = 2$.

11.4 The phase transition

Nature of the phase transition in the random field Ising model is still a mystery. The only reliable fact about it is that the upper critical dimensionality (the dimensionality above which the critical phenomena are described by the mean-field theory, Section 7.1) for this phase transition is equal to 6 (unlike pure systems where it is equal to 4). Let us consider this point in some more details.

Near the phase transition the random field Ising model can be described in terms of the scalar field Ginzburg-Landau Hamiltonian with the double-well potential:

$$H = \int d^D x \left[\frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}\tau\phi^2(x) - h(x)\phi(x) + \frac{1}{4}g\phi^4(x) \right] \tag{11.23}$$

where quenched random fields $h(x)$ are assumed to be described by the symmetric Gaussian distribution with the mean square equal to h_0^2. Ground state configurations of the fields $\phi(x)$ are defined by the saddle-point equation:

$$-\Delta\phi(x) + \tau\phi(x) + g\phi^3(x) = h(x) \tag{11.24}$$

In the usual RG approach for the phase transition in the pure systems ($h(x) = 0$) one constructs the perturbation theory over large-scale deviations on the background homogeneous solution of the above equation, $\phi_0 = \sqrt{|\tau|/g}, \quad \tau < 0$ or $\phi_0 = 0, \quad \tau > 0$ (Section 7.4). Apparently, the solutions of the equation (11.24) with nonzero $h(x)$ essentially depend on a particular configuration of the quenched fields being non-homogeneous. Let us estimate the conditions under which the external fields become the dominant factor for the ground state configurations.

Let us consider a large region Ω_L of a linear size $L >> 1$. An average value of the field in this region can be defined as follows:

$$h(\Omega_L) = \frac{1}{L^D}\int_{x\in\Omega_L} d^D x h(x) \tag{11.25}$$

Correspondingly, for the characteristic value of the field $h(\Omega_L)$ (averaged over realizations) one gets:

$$h_L \equiv \sqrt{\overline{h^2(\Omega_L)}} = \frac{1}{L^{2D}}\sqrt{\int_{x,x'\in\Omega_L} d^D x d^D x' \overline{h(x)h(x')}} = \frac{h_0}{L^{D/2}} \tag{11.26}$$

The average value of the order parameter ϕ in a given region Ω_L can be estimated from the equation:

$$\tau\phi + g\phi^3 = h_L \tag{11.27}$$

The solutions of this equation are:

$$\phi \simeq \phi_0 + \frac{h_L}{2\tau}, \quad if \quad h_L << \tau^{3/2} \tag{11.28}$$

$$\phi \simeq (\frac{h_L}{g})^{1/3}, \quad if \quad h_L >> \tau^{3/2} \tag{11.29}$$

In the first case, eq.(11.28), the external fields can be considered as small perturbations, whereas in the second case, eq.(11.29) the external fields are the dominant factor and the solution for the order parameter does not depend on the temperature parameter τ. Now let us estimate up to which characteristic sizes of the clusters the external fields could dominate. According to (11.26) the condition $h(\Omega_L) >> \tau^{3/2}$, eq.(11.29), yields:

$$L << \frac{h_0^{2/D}}{\tau^{3/D}} \tag{11.30}$$

On the other hand, the estimation of the order parameter in terms of the equilibrium equation (11.27) could be correct only at scales much greater than the size of the fluctuation region, which is equal to the correlation length $R_c \sim \tau^{-\nu}$. Thus, one has the lower bound for L:

$$L >> \tau^{-\nu} \tag{11.31}$$

Therefore the situation when the external fields become the dominant factor could exist in the region of parameters defined by the condition:

$$\tau^{-\nu} << \frac{h_0^{2/D}}{\tau^{3/D}} \tag{11.32}$$

or

$$\tau^{3-\nu D} << h_0^2 \tag{11.33}$$

Such region of temperatures near T_c exists only if:

$$\nu D < 3 \tag{11.34}$$

In this case the temperature interval near T_c in which the order parameter configurations are defined mainly by the random fields is:

$$\tau_*(h_0) \sim h_0^{\frac{2}{3-\nu D}} \tag{11.35}$$

Outside this interval, $\tau >> \tau_*$ the external fields can be considered as small perturbations to the usual critical phenomena.

In the mean field theory (which correctly describes the phase transition in the pure system for $D > 4$) $\nu = 1/2$. Thus, according to the condition (11.34) the above non-trivial temperature interval τ_* exists only at dimensions $D < 6$. Correspondingly, at dimensions $D > 6$ the phase transition is correctly described by the usual mean-field theory.

What is going on in the close vicinity of the phase transition point, $\tau << \tau_*(h_0)$, at dimensions $D < 6$ is not known. The only concrete statement for the critical behavior in the random field D-dimensional Ising model worked out some years ago claims that its critical exponents coincide with those of the pure $(D-2)$-dimensional system [74]. Unfortunately, although it is very elegant, this statement is wrong for the reasons mentioned in Section 11.2.

Indeed, let us turn back to the order parameter saddle-point equation (11.24). There exist strong indications both theoretical [79],[76],[77] and numerical [80] in favor of the possibility of the existence of many (macroscopic number) solution of this equation. Moreover, according to the numerical studies [80] there exists another critical temperature T_* above T_c such that at temperatures $T > T_*$ the solution of the saddle-point equation (11.24) is unique (this region corresponds to the usual paramagnetic phase), while at $T < T_*$ multiple solutions appear, and only below T_c the onset of the long range magnetic order takes place. All these solutions must essentially depend on a particular configuration of the quenched fields being non-homogeneous. In such a situation the usual RG approach, at least in its traditional form (which is nothing else but the perturbation theory), can not be used.

It seems probable that we could find here again a kind of a completely new type of critical phenomena of the spin-glass nature similar to that discussed in Chapter 9. As in spin-glasses [1],[2] one could find here numerous disorder dependent local energy minima. Unlike in spin-glasses, however, these minima are most probably separated by *finite* energy barriers. Therefore, it is hardly possible to expect the existence of the real spin-glass phase near T_c. Nevertheless, it is widely believed that there must be a kind of a "glassy" phase in a finite temperature interval, which separates the paramagnetic state at high temperatures from the ferromagnetic one at low temperatures [81],[82].

In the situation when the thermodynamics is defined by numerous disorder-dependent local energy minima the most developed technique, which makes it possible to perform actual calculations, is the Parisi replica symmetry breaking (RSB) scheme (Chapters 3 and 9). It is now many years since the possibility of the RSB in the random field Ising systems was first discussed [82], [83]. Recently the RSB technique has been successfully applied for the statistics of random manifolds [31], as well as for the m-component ($m >> 1$) spin systems with random fields [32]. In the last case it has been rigorously proved that the usual scaling replica-symmetric solution is unstable with respect to the RSB in the phase transition point. Moreover, recent studies of the D-dimensional random field Ising systems, made in terms of the Legendre transforms and the general scaling arguments, demonstrate that for

$D < 6$ in a finite temperature interval near T_c a new type of the critical regime is established, which is characterized by explicit RSB in the scaling of the correlation functions [84].

Although at the present state of knowledge is this field it would be very difficult to hypothesize what could be the systematic approach to the problem, one of the possibilities is that the calculations could still be done in a framework of the RG theory, in which the existing numerous solutions are selfconsistently taken into account in terms of the explicit RSB in the parameters of the renormalized Hamiltonian.

12 Conclusions

In this part of the Course we have considered the problem of the effects produced by weak quenched disorder in statistical spin systems. The idea was to demonstrate on qualitative rather than quantitative level the existing basic theoretical approaches and concepts. That is why the considerations were restricted by the simplest statistical models, and most of the details of the theoretical and experimental studies were left apart.

The key problem which still remains unsolved, is whether or not the obtained strong coupling phenomena in the RG flows could be interpreted as the onset of a kind of the spin-glass phase in a narrow temperature interval near T_c. In spin-glasses it is generally believed that RSB phenomenon can be interpreted as a factorization of the phase space into the (ultrametric) hierarchy of "valleys", or local minima pure states, separated by macroscopic (infinite) barriers. Although in the systems considered here the local minima configurations responsible for the RSB are not likely to be separated by infinite barriers, it would be natural to interpret phenomena obtained as effective factorization of the phase space into a hierarchy of valleys separated by *finite* barriers. Since the only relevant scale in the critical region is the correlation length, the maximum energy barriers must be proportional to $R_c^D(\tau)$, and they become divergent as the critical temperature is approached. In this situation one could expect that besides the usual critical slowing down (corresponding to the relaxation inside one valley) qualitatively much greater (exponentially large) relaxation times would be required for overcoming barriers separating different valleys. Therefore, the traditional measurements (made at finite equilibration times) can actually correspond to the equilibration within one valley only, and not to the true thermal equilibrium. Then in a close vicinity of the critical point different measurements of the critical properties of, for example, spatial correlation functions (in the same sample) would exhibit different results, as if the state of the system becomes effectively "trapped" in different valleys. In any case this phenomenon clearly demonstrates the existence of numerous metastable states forming infinite continuous spectrum, and it could be interconnected with a general idea that the critical phenomena should be described in terms of an infinite hierarchy of the correlation lengths and critical exponents. Unfortunately at the present state of knowledge in this field it is very difficult to hypothesise what the systematic approach for solving this type the problem should be .

It is now many years since, after the works of L.D.Landau and K.G.Wilson, the theory of the second-order phase transitions has become quite respectable and well established science. It is generally believed that no bright qualitative breakthrough can be expected in this field any more, and that the only remaining problems are more and more exact calculations of the critical exponents. In a sense, the theory of the disorder-induced critical phenomena has tried to attain a similar status. However, recent developments in this field clearly indicate the existence of a qualitatively new physical phenomena, which goes well beyond the traditional concepts of the scaling theory. It seems as if we are close to a breakthrough to a new level of understanding of the critical phenomena in weakly disordered materials. I do believe so. This is in fact the main reason why the present review has been written.

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Figure Captures

Fig.1. Free energy of the ferromagnetic Ising magnet:
(a) in the zero external magnetic field;
(b) in non-zero magnetic field.

Fig.2. The frustrations in the system of three spins:
(a) No frustration: the product of the interactions along the triangle is positive.
(b) The frustrated triangle: the product of the interactions along the triangle is negative.

Fig.3. The qualitative structure of the spin-glass free energy landscape at different temperatures.

Fig.4. The probability distribution function $P(q)$:
(a) in the paramagnetic phase;
(b) in the ferromagnetic phase;
(c) in the spin glass phase.

Fig.5. The hierarchical tree of the spin glass states.

Fig.6. The structure of the matrix Q_{ab} at the one-step replica symmetry breaking.

Fig.7. The grouping of replicas at the two-steps replica symmetry breaking.

Fig.8. The tree-like definition of the matrix elements Q_{ab} for the two-steps RSB.

Fig.9. The explicit form of the matrix Q_{ab} for the two-steps RSB.

Fig.10. The qualitative shape of the functions $q(x)$ and $P(q)$:
(a) in the zero magnetic field near the critical point ($\tau << 1$);
(b) in finite magnetic field h, for $0 < h < h_c(T)$ and $\tau << 1$;
(c) in the zero magnetic field and in the limit of low temperatures, $T << 1$.

Fig.11. The ultrametric tree of the spin-glass states.

Fig.12. The relaxation behaviour of the magnetization in the field cooled aging experiments.

Fig.13. The relaxation behaviour of the magnetization in the zero field cooled aging experiments.

Fig.14. The relaxation behaviour of the magnetization in the aging experiments with the cooling temperature cycles.

Fig.15. The relaxation behaviour of the magnetization in the aging experiments with the heating temperature cycles.

Fig.16. The relaxation behaviour of the magnetization at the temperature T after the aging at the temperature $T - \Delta T$.

Fig.17. The dependence of the values of the free energy barriers at the temperature T from their values at the temperature $T - \Delta T$.

Fig.18. The dependence of $d\Delta/dT$ from the values of the barriers Δ.

Fig.19. Diagrammatic representation of the interaction energy $V[\tilde{\phi}, \varphi]$.

Fig.20. Diagrammatic representation of the first order perturbation contribution $\langle V \rangle$.

Fig.21. Diagrammatic representation of the second order perturbation contribution $\langle\langle V^2 \rangle\rangle$.

Fig.22. (a) Diagrammatic representation of the specific heat.
(b) The diagram which contribute to the renormalization of the "dressed" mass $m(\xi)$.

Fig.23. Diagrammatic representation the interaction term $g_{ab}(\phi_i^a(x))^2(\phi_j^b(x))^2$.

Fig.24. The diagrams which contribute to the interaction terms $g_{ab}(\phi_i^a(x))^2(\phi_j^b(x))^2$.

Fig.25. The diagrams which contribute to the renormalization of the "mass" term $\tau(\phi_i^a(x))^2$.

Fig.26. Lattice graphs of the high temperature expansion of the 2D Ising model.

Fig.27. Diagrammatic representation the interaction term $u(\overline{\psi^a}(x)\psi^a(x))(\overline{\psi^b}(x)\psi^b(x))$ and the mass term $\tau(\overline{\psi^a}(x)\psi^a(x))$.

Fig.28. The diagrams which contribute to interaction term $u(\overline{\psi^a}(x)\psi^a(x))(\overline{\psi^b}(x)\psi^b(x))$.

Fig.29. The diagrams which contribute to the mass term $\tau(\overline{\psi^a}(x)\psi^a(x))$.

Fig.30. The specific heat C at the critical temperature plotted as a function of $\ln L$:
(1) the exact asymptotic result for the pure system, $r = 1$;
(2) $r = 1/2$ with fitting parameters $C_0 = 0.048, C_1 = 15.7, b = 0.085$;
(3) $r = 1/4$ with fitting parameters $C_0 = 0.048, C_1 = 2.04, b = 0.35$;
(4) $r = 1/10$ with fitting parameters $C_0 = -0.28, C_1 = 0.224, b = 8.8$.

Fig.31. The same set of data as in Fig.30, plotted against $\ln\ln L$.

Fig.32. The same set of data as in Fig.30 for $r = 1/4$, plotted against $\ln(1 + b\ln L)$ with $b = 0.35$.

Fig.33. A naive phase diagram of a ferromagnetic system diluted by antiferromagnetic or broken couplings.

Fig.34. Phase diagram of the Ising ferromagnet diluted by antiferromagnetic couplings; $T_N(u)$ is the Nishimory line.